

## Automatic Control and Flight Control

Jörg J. Buchholz jjbuchholz.de

John Cater profiles.canterbury.ac.nz/John-Cater

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# Nomenclature

angle of attack
subjective weighting factor
dynamic pressure
side-slip angle
flight-path azimuth
ideal impulse (Dirac impulse)
stick deflection (pull)
stick deflection (to the right)
pedal deflection
elevator deflection
flight-path (inclination) angle, angle of climb
imaginary part of a complex number
Laplace operator
inverse Laplace operator
Nabla operator
angular frequency
natural angular frequency
corner frequency
centre frequency
conjugate quaternion
real part of a complex number
air density
time (as integration variable)

ε	angle of the pole with respect to the imaginary axis
ε	infinitesimal small value
$\Phi$	roll angle, bank angle
$\varphi$	phase angle, phase shift
$\Psi$	yaw angle
Θ	pitch angle
Ω	angular velocity vector
${\it \Phi}$	attitude vector (Euler angle vector)
$\boldsymbol{A}$	system matrix, state matrix
В	input matrix, control matrix
C	output matrix, measurement matrix
D	angular momentum vector
D	feedforward matrix, direct transmission matrix
f	vector differential equation
G	weight vector
g	vector output equation
Ι	inertia tensor
$M_{fa}$	transformation matrix from the aerodynamic to the body-fixed axis system
$M_{fg}$	transformation matrix from the geodetic to the body-fixed axis system
$M_{kg}$	transformation matrix from the geodetic to the flight-path axis system
$oldsymbol{M}_\Omega$	quaternion differential equation matrix
$\boldsymbol{n}$	axis of rotation vector of a quaternion
P	momentum vector
${old Q}$	moment vector
$r_F$	distance of the engine from the reference point
R	resulting force vector
8	position vector (direction, distance)
$\boldsymbol{u}$	input vector
V	velocity vector

$\boldsymbol{v}$	output vector
x	state vector
Ξ	angle of rotation of a quaternion
ξ	aileron deflection
$\zeta$	rudder deflection
A	amplitude (ratio)
A	area
A	lift
a	real part of a quaternion
b, c, d	imaginary parts of a quaternion
С	spring constant
$C_A$	lift coefficient
$C_l$	roll moment coefficient
$C_m$	pitch moment coefficient
$C_n$	yaw moment coefficient
$C_Q$	side force coefficient
$C_W$	drag coefficient
$C_{A\alpha}$	lift due to angle of attack etc.
$C_{W0}$	zero drag
D	damping
$d\left(t\right)$	square wave pulse
dB	dezibel
E	aerodynamic force unit
$e\left(t ight)$	control error, control difference, deviation
$e_{max}$	maximum overshoot
F	force
f	frequency
$f\left(t ight)$	ramp response in time domain
$F_c$	thrust reference

g	gravitational acceleration
$G\left(s\right)$	transfer function
$g\left(t ight)$	impulse response in time domain
$G\left(z ight)$	transfer function in the z-domain
$G_0$	open loop transfer function
$G_g$	overall transfer function
$G_M$	sensor transfer function
$G_R$	controller transfer function
$G_S$	plant transfer function
$G_V$	forward transfer function
$G_V$	open loop controller transfer function
$G_z$	disturbance transfer function
$G_{ew}$	reference control error
$G_{ez}$	disturbance control error
$G_{St}$	disturbance feedforward transfer function
$H\left(s ight)$	step response in frequency domain
$h\left(t ight)$	step response in time domain
Ι	integral criteria
i,j,k	imaginary units of a quaternion (or of a complex number)
$I_x, I_y, I_z$	moments of inertia
$I_{xz}$	product of inertia
K	gain
k	integer number
$K_{\beta}$	side-slip angle controller
$K_{\chi}$	flight-path azimuth controller
$K_{\Phi}$	bank angle controller
$K_{\Theta}$	pitch controller
$K_H$	altitude controller
$K_H$	auxiliary variable

$K_R$	controller gain
$K_V$	airspeed controller, autothrottle
$K_{\eta q}$	pitch damper
$K_{\xi p}$	roll damper
$K_{\zeta r}$	yaw damper
$K_{R_{krit}}$	amplitude reserve, critical gain
L	roll moment
$l_{\mu}$	mean aerodynamic chord
M	pitch moment
m	mass
Ma	Mach number
N	yaw moment
p	roll speed
$p_A^*$	normalised aerodynamic roll speed etc.
Q	side force
q	pitch speed
r	damping factor
r	magnitude of a complex number
r	yaw speed
$r\left(t\right)$	unit ramp
s	Laplace variable
$S\left(s ight)$	unit step in frequency domain
$s\left(t ight)$	unit step in time domain
Т	period
Т	sampling time
t	time
$t_{\varepsilon}$	magnitude of the control error stays less than $\varepsilon$
$T_D$	D-element time constant
$T_F$	engine time constant

$T_g$	compensation time
$T_I$	integrator time constant
$T_N$	normalisation time constant
$T_N$	reset time
$T_T$	dead time constant
$T_u$	delay time
$T_V$	reset time
$t_{an}$	response time
$T_{krit}$	period of the critical oscillation
u	forward speed
$U\left(s ight)$	input variable in frequency domain
$u\left(t ight)$	input variable in time domain
$u_k$	input variable in z-domain
v	side speed
$V\left(s ight)$	output variable in frequency domain
$v\left(t ight)$	output variable in time domain
$V_A$	aerodynamic speed, air speed
$v_k$	output variable in z-domain
W	drag
w	sink speed
$w\left(t ight)$	reference input variable
X	<b>x</b> component of the force vector (towards the front)
x	state variable
x	<b>x</b> component of the position vector (towards the front)
$x\left(t ight)$	controlled variable
Y	y component of the force vector (towards the right)
y	y component of the position vector (towards the right)
$y\left(t ight)$	manipulated variable
Ζ	quaternion

Ζ	z component of the force vector (towards the bottom)
z	complex number
z	independent variable of the transfer function in the z-domain
z	z component of the position vector (towards the bottom)
$z\left(t ight)$	disturbance
$Z^0$	unit quaternion
$Z_D$	unit quaternion of rotation
$Z_{\Omega}$	flight-path angular velocity quaternion
Index $A$	aerodynamic
Index $a$	aerodynamic axis system
Index $F$	thrust
Index $f$	body-fixed axis system
Index $g$	geodetic axis system, Earth-fixed axis system
Index $K$	flight-path
Index $k$	flight-path axis system
Index $W$	wind
PFD	partial fraction decomposition

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# Part I

# **Fundamentals of Automatic Control**

## Chapter 1

## Introduction

## 1.1 Block Diagrams and Terminology

#### 1.1.1 Standards

DIN 19221	Control technology; letter symbols
DIN 19225	Naming and classification of controllers
DIN 19226	Control Engineering, Definitions and terms
VDI/VDE 3526	Designations for control and regulating circuits

#### 1.1.2 Signal Connections



Figure 1.1: Signal connections.

#### 1.1.3 Blocks



Figure 1.2: Differential equation.



Figure 1.3: Step response.



Figure 1.4: Transfer function.

#### 1.1.4 Terminology

Table 1.1: Variables	for Automatic	Control
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u	input variable
v	output variable
w	reference input variable, command, setpoint variable
x	control(led) variable
e	control error, control difference, deviation, error signal,
y	manipulated variable, correcting variable
z	disturbance (variable)
s	Laplace variable, complex frequency
$G\left(s ight)$	transfer function

## 1.2 Open Loop and Closed Loop Control

#### 1.2.1 Open Loop Control



Figure 1.5: Open loop control.

• Open sequence of effects, no feedback

- Can only compensate for known disturbances
- Cannot become unstable
- Fast

#### 1.2.2 Closed Loop Control



Figure 1.6: Closed loop control.

- Closed control loop
- Can also compensate for unknown disturbances
- May become unstable
- Slower

## **1.3** Special Excitation Functions and System Response

#### 1.3.1 Step





Figure 1.7: Unit step.

#### 1.3.2 Ramp





Figure 1.8: Unit ramp.

#### 1.3.3 Square Wave Pulse

$$d\left(t\right) = \begin{cases} 1/\varepsilon & \text{if } 0 \leq t \leq \varepsilon \\ 0 & \text{otherwise} \end{cases}$$



Figure 1.9: Unit square wave pulse.

#### 1.3.4 Ideal Impulse (Dirac Impulse)

$$\delta(t) = \lim_{\varepsilon \to 0} d(t) = \begin{cases} \infty & \text{if } t = 0\\ 0 & \text{otherwise} \end{cases}$$



Figure 1.10: Ideal impulse (Dirac impulse).

#### 1.3.5 Sine





Figure 1.11: Unit sine.

## 1.3.6 Relation between Ramp, Step and Impulse

$$r(t) \xrightarrow[\int dt]{\frac{\partial}{\partial t}} s(t) \xrightarrow[\int dt]{\frac{\partial}{\partial t}} \delta(t)$$

#### 1.3.7 System Response



Figure 1.12: System responses.

## 1.4 Static and Dynamic Behaviour



Figure 1.13: General (static or dynamic) system.

#### 1.4.1 Static Behaviour (Static Characteristic Curve)

- Example: ideal measuring amplifier
- Change in the output variable only if the input variable is currently changing
- No energy storage  $\rightarrow$  no self-movement (no intrinsic dynamics)
- Description using algebraic equation

#### 1.4.2 Dynamic Behaviour

- Example: pendulum
- Change in the output variable without the input variable currently changing
- Internal energy storage is charged and discharged  $\rightarrow$  self-movement (intrinsic dynamics)
- Description using differential equations



Figure 1.14: Dynamic Behaviour.

### 1.5 Linear and Non-Linear Behaviour

In many ways, linear systems are easier to use than non-linear systems. They are generally much easier to analyse, control and simulate. To check whether a general nonlinear system of the form v = g(u) is linear, two linearity conditions are used, both of which must be fulfilled:



Figure 1.15: Gain principle:  $g(K \cdot u) \stackrel{!}{=} K \cdot g(u)$ .



Figure 1.16: Superposition principle:  $g(u_1 + u_2) \stackrel{!}{=} g(u_1) + g(u_2)$ .

 $\pmb{\times}$  "If it works for one input amplitude, it always works."

**X** The order of linear blocks is interchangeable: g(h(u)) = h(g(u))

#### 1.5.1 Linear Blocks

- Integrator
- Differentiator
- Gain (constant)
- Sum
- Time delay
- Composite blocks: PT1, PT2, PD-T<sub>1</sub>-filter, PID-controller,  $\ldots$
- General transfer function
- Linear state space representation

## Chapter 2

## Linear Systems

## 2.1 Second-Order Mechanical Oscillator



Figure 2.1: Second-order mechanical oscillator (v: displacement).



Figure 2.2: Free-body second-order oscillator.

Second-order linear inhomogeneous differential equation:

$$m\ddot{v}+r\dot{v}+cv=F$$

Normal form:

$$\ddot{v} + \frac{r}{m}\dot{v} + \frac{c}{m}v = \frac{F}{m}$$

General second-order system:

$$\ddot{v} + 2D\omega_0\dot{v} + \omega_0^2v = K\omega_0^2u$$

Input variable:

$$u = F$$

Natural angular frequency:

$$\omega_0^2 = \frac{c}{m} \quad \Rightarrow \quad \omega_0 = \sqrt{\frac{c}{m}}$$

Damping:

$$2D\omega_0 = \frac{r}{m} \quad \Rightarrow \quad D = \frac{r}{2m\omega_0} = \frac{r}{2m\sqrt{\frac{c}{m}}} = \frac{r}{2\sqrt{cm}}$$

Gain factor:

$$K\omega_0^2 = \frac{1}{m} \quad \Rightarrow \quad K = \frac{1}{m\omega_0^2} = \frac{1}{m\frac{c}{m}} = \frac{1}{c}$$

$$\begin{array}{c} u(t) \\ \hline \\ \hline \\ \dot{v} + 2D\omega_0\dot{v} + \omega_0^2v = K\omega_0^2u \end{array} \begin{array}{c} v(t) \\ \hline \\ \hline \end{array}$$

Figure 2.3: Block diagram of second-order oscillator.

#### 2.1.1 Impulse Response

Impulse as input variable:  $u(t) = \delta(t)$ 



g(t) D = 1 (b) D = 1

Figure 2.4: Impulse response g(t).

#### 2.1.2 Step Response

Step as input variable: u(t) = s(t)







## 2.2 Laplace Transform (of a PT2)

Differential equation:

$$\ddot{v} + 2D\omega_0\dot{v} + \omega_0^2v = K\omega_0^2u$$

Initial values are equal to zero:

 $v(0) = \dot{v}(0) = 0$ 



Figure 2.6: Solution of a differential equation using Laplace transform.

Laplace transform:

- of the input function:  $\mathcal{L} \{ u(t) \} = U(s)$
- of the output function:  $\mathcal{L} \{ v(t) \} = V(s)$
- of the first derivative of the output function:  $\mathcal{L}\left\{\dot{v}\left(t\right)\right\} = s \cdot V\left(s\right)$
- of the second derivative of the output function:  $\mathcal{L}\left\{\ddot{v}\left(t\right)\right\} = s^2 \cdot V\left(s\right)$

Transformed differential equation:

$$s^{2}V(s) + 2D\omega_{0}sV(s) + \omega_{0}^{2}V(s) = K\omega_{0}^{2}U(s)$$

Transfer function:

$$G(s) = \frac{V(s)}{U(s)} = \frac{K\omega_0^2}{s^2 + 2D\omega_0 s + \omega_0^2}$$

### 2.2.1 Table of some Laplace Transforms

$x\left(t ight)$	$\mathcal{L}\left\{ x\left( t\right) \right\} =X\left( s\right)$
$\delta\left(t\right)$	1
$s\left(t\right) = 1$	$\frac{1}{s}$
$r\left(t\right) = t$	$\frac{1}{s^2}$
$e^{-at}$	$\frac{1}{s+a}$
$e^{-at}\sin\omega t$	$\frac{\omega}{(s\!+\!a)^2\!+\!\omega^2}$
$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Table 2.1: The most important Laplace transforms.

#### 2.2.2 Example: Step Response of a PT2

Transfer function:

$$G(s) = \frac{1}{s^2 + s + 1}$$
 ( $\omega_0 = 1, K = 1, D = 0.5$ )

Step:

$$\mathcal{L}\left\{s\left(t\right)\right\} = S\left(s\right) = \frac{1}{s}$$

Step response:

$$H(s) = G(s) \cdot S(s) = \frac{1}{s^2 + s + 1} \cdot \frac{1}{s}$$

Partial fraction decomposition:

$$H\left(s\right) = \frac{As+B}{s^2+s+1} + \frac{C}{s}$$

Same denominators:

$$\frac{1}{(s^2+s+1)s} = \frac{(As+B)s + C(s^2+s+1)}{(s^2+s+1)s}$$

Coefficient comparison in the numerator:

$$s^{0}: \quad 1 = C \quad \Rightarrow \quad C = 1$$
  

$$s^{1}: \quad 0 = B + C = B + 1 \quad \Rightarrow \quad B = -1$$
  

$$s^{2}: \quad 0 = A + C = A + 1 \quad \Rightarrow \quad A = -1$$

Result of the partial fraction decomposition:

$$H(s) = \frac{1}{s} - \frac{s+1}{s^2 + s + 1}$$

Completing the square:

$$H\left(s\right) = \frac{1}{s} - \frac{s+1}{\left(s+\frac{1}{2}\right)^2 + 1 - \frac{1}{4}} = \frac{1}{s} - \frac{s+1}{\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}}$$

Comparison with Laplace transform table (table 2.1):

$$a = \frac{1}{2} \quad \omega = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

Adapt numerator to table by splitting:

$$H(s) = \frac{1}{s} - \frac{s + \frac{1}{2}}{\left(s + \frac{1}{2}\right)^2 + \frac{3}{4}} - \frac{\frac{1}{2}}{\left(s + \frac{1}{2}\right)^2 + \frac{3}{4}}$$

Adapt the third addend to the table:

$$H(s) = \frac{1}{s} - \frac{s + \frac{1}{2}}{\left(s + \frac{1}{2}\right)^2 + \frac{3}{4}} - \frac{\frac{1}{2}\sqrt{\frac{3}{4}}\sqrt{\frac{4}{3}}}{\left(s + \frac{1}{2}\right)^2 + \frac{3}{4}}$$

Ready to transform back:

$$H(s) = \frac{1}{s} - \frac{s + \frac{1}{2}}{\left(s + \frac{1}{2}\right)^2 + \frac{3}{4}} - \frac{1}{\sqrt{3}} \frac{\sqrt{\frac{3}{4}}}{\left(s + \frac{1}{2}\right)^2 + \frac{3}{4}}$$

Step response in the time domain:

$$h(t) = \mathcal{L}^{-1} \{ H(s) \} = 1 - e^{-\frac{1}{2}t} \cos \sqrt{\frac{3}{4}t} - \frac{1}{\sqrt{3}} e^{-\frac{1}{2}t} \sin \sqrt{\frac{3}{4}t}$$
(2.1)

Period:



Figure 2.7: Step response of a second-order system (D = 0.5).

## 2.3 Limit Theorems

Initial value theorem:

$$f(t = 0) = \lim_{t \to 0} f(t) = \lim_{s \to \infty} s \cdot F(s)$$

Final value theorem:

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s \cdot F(s)$$

#### 2.3.1 Example: Step Response of a PT2

General (final value theorem):

$$\lim_{t \to \infty} h(t) = \lim_{s \to 0} s \cdot H(s) = \lim_{s \to 0} s \cdot G(s) \cdot \frac{1}{s} = \lim_{s \to 0} G(s)$$

Special (PT2):

$$\lim_{t \to \infty} h(t) = \lim_{s \to 0} G(s) = \lim_{s \to 0} \frac{K\omega_0^2}{s^2 + 2D\omega_0 s + \omega_0^2} = \frac{K\omega_0^2}{\omega_0^2} = K$$

## 2.4 Poles of the Transfer Function

Example: Second-order system (PT2)

Transfer function:

$$G\left(s\right) = \frac{K\omega_{0}^{2}}{s^{2} + 2D\omega_{0}s + \omega_{0}^{2}}$$

Poles are zeros of the denominator:

$$s^2 + 2D\omega_0 s + \omega_0^2 = 0$$

Two (real or complex conjugate) poles:

$$s_{1,2} = -D\omega_0 \pm \sqrt{D^2 \omega_0^2 - \omega_0^2} = -D\omega_0 \pm \omega_0 \sqrt{D^2 - 1}$$

Case distinction:

Table $2.2$ :	Position	of the	Poles	Depending	on the	Damping
---------------	----------	--------	-------	-----------	--------	---------

D  > 1	$s_{1,2}=-D\omega_0\pm\omega_0\sqrt{D^2-1}$	two real poles
D  < 1	$s_{1,2} = \underbrace{-D\omega_0}_{\sigma} \pm j \underbrace{\omega_0 \sqrt{1 - D^2}}_{\omega}$	complex conjugate pair of poles
D = 0	$s_{1,2} = \pm \mathbf{j}\omega_0$	complex conjugate pair of poles on the imaginary axis
D = 1	$s_{1,2} = -\omega_0$	real double pole on the left half-plane
D = -1	$s_{1,2} = \omega_0$	real double pole on the right half-plane (unstable)



Figure 2.8: Pole distribution of a second-order system.

Stability: "A system with at least one pole in the right half-plane is unstable." Natural frequency is distance from the origin:

$$|\sigma|^{2} + \omega^{2} = |-D\omega_{0}|^{2} + \left(\omega_{0}\sqrt{1-D^{2}}\right)^{2} = D^{2}\omega_{0}^{2} + \omega_{0}^{2}\left(1-D^{2}\right) = \omega_{0}^{2}$$

"The closer to the origin, the slower."

Relationship between angle  $\varepsilon$  and damping D (in the second quadrant):

$$\sin \varepsilon = \frac{|\sigma|}{\omega_0} = \frac{D\omega_0}{\omega_0} = D$$

"The closer to the imaginary axis, the worse the damping"

### 2.5 Frequency Response

Sine input:

$$u(t) = A_u \sin \omega t$$

$$u(t) = A_u \cdot \sin(\omega t)$$
Linear system
$$v(t) = A_v \cdot \sin(\omega t + \phi)$$

Figure 2.9: Sine input and response of a linear system.

Steady-state sine response (transient process completed) has:

- same frequency  $\omega$
- different amplitude  $A_v$
- different phase (phase shift)  $\varphi$



Figure 2.10: Sine input and response of a linear system.

The frequency response consists of the amplitude response (ratio of output to input amplitude over frequency) and the phase response (phase shift depending on frequency):



Figure 2.11: Amplitude response.



Figure 2.12: Phase response.

## 2.6 Bode Diagram and Nyquist Plot

#### 2.6.1 Properties of Complex Numbers

Complex number in Cartesian form:

$$z = a + jb$$

Complex number in exponential form:

$$z = r \cdot e^{\mathbf{j}\varphi}$$



Figure 2.13: Point representation of the complex number z.

Magnitude:

$$|z| = r = \sqrt{a^2 + b^2}$$

Phase:

$$\varphi = \arctan\left(\frac{b}{a}\right)$$

Real part:

$$\Re\left(z\right) = a = r \cdot \cos\left(\varphi\right)$$

Imaginary part:

$$\Im(z) = b = r \cdot \sin(\varphi)$$

Quotient of two complex numbers:

$$z_Q = \frac{r_1 \cdot e^{j\varphi_1}}{r_2 \cdot e^{j\varphi_2}} = \frac{a + jb}{c + jd}$$

Magnitude equals the quotient of the individual magnitudes:

$$|z_Q| = \frac{r_1}{r_2} = \frac{\sqrt{a^2 + b^2}}{\sqrt{c^2 + d^2}}$$

Phase equals the difference between the individual phases:

$$\varphi_Q = \varphi_1 - \varphi_2 = \arctan\left(\frac{b}{a}\right) - \arctan\left(\frac{d}{c}\right)$$

The real and imaginary parts result from complex conjugate expansion:

$$z_Q = \frac{a+jb}{c+jd} = \frac{a+jb}{c+jd} \cdot \frac{c-jd}{c-jd} = \frac{ac-ajd+jbc-jbjd}{cc-cjd+jdc-jdjd} = \frac{ac+bd+j(bc-ad)}{c^2+d^2}$$

Real part:

$$\Re\left(z_Q\right) = \frac{ac + bd}{c^2 + d^2}$$

Imaginary part:

$$\Im\left(z_Q\right) = \frac{bc - ad}{c^2 + d^2}$$

#### 2.6.2 Logarithmic Scaling (Decibels)

Amplitude response double logarithmic scaling: logarithmic frequency and amplitude in decibels

Phase response single-logarithmic scaling: logarithmic frequency only

Converting decibels:

• 
$$A[dB] = 20 \cdot \log_{10} A$$

• 
$$A = 10^{\frac{A[dB]}{20}}$$

 Table 2.3:
 Some Conversion Examples

A[dB]	0	20	-40	80	6	3	$-\infty$
A	1	10	0.01	10000	$\approx 2$	$\approx \sqrt{2}$	0

## 2.6.3 Example: Second-Order System (PT2)

Transfer function:

$$G(s) = \frac{0.1}{s^2 + s + 1}$$

Frequency response:

$$G(j\omega) = \frac{0.1}{(j\omega)^2 + j\omega + 1} = \frac{0.1}{(1 - \omega^2) + j\omega}$$
(2.2)  
$$= \frac{0.1}{(1 - \omega^2) + j\omega} \cdot \frac{(1 - \omega^2) - j\omega}{(1 - \omega^2) - j\omega} = \frac{0.1 \cdot (1 - \omega^2 - j\omega)}{(1 - \omega^2)^2 + \omega^2}$$
(2.3)

The amplitude response and the phase response follow from eqn. (2.2): Amplitude response:

$$A(j\omega) = |G(j\omega)| = \frac{0.1}{\sqrt{(1-\omega^2)^2 + \omega^2}}$$

Phase response:

$$\varphi\left(\omega\right) = -\arctan\left(\frac{\omega}{1-\omega^2}\right)$$



Figure 2.14: Bode diagram: Amplitude response in decibels and phase response over logarithmic frequency.

The real part and the imaginary part follow from eqn. (2.3): Real part:

$$\Re (G (j\omega)) = \frac{0.1(1-\omega^2)}{(1-\omega^2)^2 + \omega^2}$$

Imaginary part:

$$\Im\left(G\left(\mathbf{j}\omega\right)\right) = \frac{-0.1\omega}{\left(1-\omega^{2}\right)^{2}+\omega^{2}}$$



Figure 2.15: Nyquist plot: Imaginary part over real part of the frequency response with frequency as the parameter of the plot.

### 2.7 P-Element

Other names: Proportional controller, gain

Example: (ideal) audio amplifier

"Differential equation" in the time domain:

$$v\left(t\right) = K \cdot u\left(t\right)$$

"Differential equation" in the frequency domain:

$$V\left(s\right) = K \cdot U\left(s\right)$$

Transfer function:

$$G\left(s\right) = \frac{V\left(s\right)}{U\left(s\right)} = K$$

Poles: none

Zeroes: none

Frequency response:

$$G\left(j\omega\right) = K$$

Amplitude response:

$$A = |G(j\omega)| = K$$

Phase response:





Figure 2.16: P-element.

Step response in the frequency domain:

$$H(s) = G(s) \cdot \mathcal{L} \{ s(t) \} = G(s) \cdot \frac{1}{s} = K \cdot \frac{1}{s}$$

Step response in the time domain (compare to table 2.1):

$$h(t) = \mathcal{L}^{-1} \left\{ H(s) \right\} = \mathcal{L}^{-1} \left\{ K \cdot \frac{1}{s} \right\} = K$$





### 2.8 PT1

Other name: Proportional control 1st-order time constant, First order low pass Example: Charging a capacitor C via a resistor R



Figure 2.18: Example of a PT1.

Differential equation in the time domain:

$$T\dot{v}(t) + v(t) = K \cdot u(t)$$

Differential equation in the frequency domain:

$$V(s) \cdot (Ts+1) = K \cdot U(s)$$

Transfer function:

$$G(s) = \frac{V(s)}{U(s)} = \frac{K}{Ts+1}$$

Poles:

$$Ts + 1 = 0 \quad \Rightarrow \quad s = -\frac{1}{T}$$

Zeroes: none



Figure 2.19: Poles (and zeros) of a (stable) PT1.

Frequency response:

$$G\left(\mathbf{j}\omega\right) = \frac{K}{\mathbf{j}\omega T + 1}$$

Amplitude response:

$$A(\omega) = |G(j\omega)| = \frac{K}{\sqrt{(\omega T)^2 + 1}}$$

Phase response:

$$\varphi\left(\omega\right) = -\arctan\omega T$$


Figure 2.20: Bode diagram of a PT1 with corner frequency  $\omega_e = 1/T$ .



Figure 2.21: Nyquist plot of a PT1.

Step response in the frequency domain::

$$H(s) = G(s) \cdot \frac{1}{s} = \frac{K}{Ts+1} \cdot \frac{1}{s} = \dots \text{PFD} \dots = \frac{K}{s} - \frac{KT}{Ts+1} = \frac{K}{s} - \frac{K}{s+\frac{1}{T}}$$

Step response in the time domain (compare table 2.1):

$$h(t) = \mathcal{L}^{-1} \{ H(s) \} = K - K e^{-\frac{t}{T}} = K \left( 1 - e^{-\frac{t}{T}} \right)$$



Figure 2.22: Step response of a PT1.



Figure 2.23: Block diagram of a PT1.

### 2.9 PT2

Other names: Proportional control 2nd-order time constant, Second-order low pass, Second-order oscillator

Example: Spring-mass oscillator (see section 2.1)

Differential equation in the time domain:

$$\ddot{v}(t) + 2D\omega_0 \dot{v}(t) + \omega_0^2 v(t) = K\omega_0^2 u(t)$$

Differential equation in the frequency domain:

$$V(s) \cdot \left(s^2 + 2D\omega_0 s + \omega_0^2\right) = K\omega_0^2 U(s)$$

Transfer function:

$$G(s) = \frac{V(s)}{U(s)} = \frac{K\omega_0^2}{s^2 + 2D\omega_0 s + \omega_0^2}$$

Poles: see section 2.4

Zeroes: none

Frequency response:

$$G(j\omega) = \frac{K\omega_0^2}{(\omega_0^2 - \omega^2) + 2D\omega_0\omega_j}$$

Amplitude response:

$$A(\omega) = |G(j\omega)| = \frac{K\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2D\omega_0\omega)^2}}$$

Phase response:

$$\varphi(\omega) = -\arctan\left(\frac{2 D\omega_0 \omega}{\omega_0^2 - \omega^2}\right)$$



Figure 2.24: Bode diagram of a PT2 with different damping values.



Figure 2.25: Nyquist plot of a PT2 with different damping values.

Step response (compare eqn. (2.1)):

$$h(t) = K\left(1 - e^{-D\omega_0 t} \left(\cos\left(\sqrt{1 - D^2}\omega_0 t\right) + \frac{D}{\sqrt{1 - D^2}}\sin\left(\sqrt{1 - D^2}\omega_0 t\right)\right)\right)$$



Figure 2.26: Step responses of a PT2 with different damping values.



Figure 2.27: Block diagram of a PT2.

## 2.10 I-Element

Other names: integrator, energy storage Example: water level in a container



Figure 2.28: Example of an I-element.

Integral equation in the time domain:

$$v(t) = \frac{1}{T_I} \int_0^t u(\tau) \, \mathrm{d}\tau + v(t=0)$$

Differential equation in the time domain:

$$\dot{v}(t) T_I = u(t)$$

Differential equation in the frequency domain:

$$V(s) s \cdot T_I = U(s)$$

Transfer function:

$$G(s) = \frac{V(s)}{U(s)} = \frac{1}{T_I \cdot s} \left( = \frac{K_I}{s} \quad \text{with} \quad K_I = \frac{1}{T_I} \right)$$

Poles:

$$T_I s = 0 \quad \Rightarrow \quad s_1 = 0$$

Zeroes: none



Figure 2.29: Poles (and zeros) of an I-element.

Frequency response:

$$G(j\omega) = \frac{1}{T_I j\omega} = -j \frac{1}{\omega T_I} = \frac{1}{\omega T_I} e^{-j\frac{\pi}{2}}$$

Amplitude response:

$$A(\omega) = |G(j\omega)| = \frac{1}{\omega T_I}$$

Phase response:

$$\varphi(\omega) = \arctan \frac{-\frac{1}{\omega T_I}}{0} = -\arctan \infty = -\frac{\pi}{2}$$



 $\bigcirc$ 



Step response in the frequency domain:

$$H(s) = G(s)\frac{1}{s} = \frac{1}{T_{I}s}\frac{1}{s} = \frac{1}{T_{I}s^{2}}$$

Step response in the time domain:

$$h(t) = \mathcal{L}^{-1} \{ H(s) \} = \mathcal{L}^{-1} \left\{ \frac{1}{T_I s^2} \right\} = \frac{t}{T_I} = \frac{1}{T_I} t$$





(b) Block diagram



## 2.11 D-Element

Other names: differentiator

Example: the angle of rotation of the speedometer needle is the derivative of the wheel angle:

$$\varphi_{speedometer} = \dot{\varphi}_{wheel}$$



Figure 2.32: Example of a D-element.

Differential equation in the time domain:

$$v\left(t\right) = T_D \dot{u}\left(t\right)$$

Differential equation in the frequency domain:

$$V\left(s\right) = T_D s U\left(s\right)$$

Transfer function:

$$G\left(s\right) = \frac{V\left(s\right)}{U\left(s\right)} = T_D s$$

Poles: none

Zeroes:



Figure 2.33: (Poles and) zeroes of a D-element.

Frequency response:

$$G\left(\mathbf{j}\omega\right) = \mathbf{j}\omega T_D = \omega T_D \mathbf{e}^{\mathbf{j}\frac{\pi}{2}}$$

Amplitude response:

$$A(\omega) = |G(\mathbf{j}\omega)| = \omega T_D$$

Phase response:

$$\varphi(\omega) = \arctan \frac{\omega T_D}{0} = \arctan \infty = \frac{\pi}{2}$$



Figure 2.34: D-element.

Step response in the frequency domain:

$$H(s) = G(s)\frac{1}{s} = T_D s \frac{1}{s} = T_D$$

Step response in the frequency domain (compare table 2.1):

 $h(t) = \mathcal{L}^{-1} \{ H(s) \} = \mathcal{L}^{-1} \{ T_D \} = T_D \delta(t) \quad \text{(Dirac impulse)}$ 







## 2.12 PID-Element

Other name: PID-controller



Figure 2.36: Parallel circuit of P-element, I-element and D-element.

Transfer function:

$$G(s) = K + \frac{1}{T_{I}s} + T_{D}s = \frac{KT_{I}s + 1 + T_{D}T_{I}s^{2}}{T_{I}s}$$

with:

$$T_I = \frac{T_N}{K}$$
 and  $T_D = T_V K$ 

$$G\left(s\right) = \frac{K\frac{T_{N}}{K}s + 1 + T_{V}K\frac{T_{N}}{K}s^{2}}{\frac{T_{N}}{K}s} = K\frac{T_{N}s + 1 + T_{V}T_{N}s^{2}}{T_{N}s} = K\left(1 + \frac{1}{T_{N}s} + T_{V}s\right)$$

with:

 $T_N$ : reset time and  $T_V$ : rate time



Figure 2.37: Hardware-oriented representation of a PID-element.

Poles:

 $s_1 = 0$ 

Zeroes:

Figure 2.38: Poles and zeros of a PID-element.

Frequency response:

$$G(j\omega) = K + \frac{1}{j\omega T_I} + j\omega T_D = K + \left(\frac{-1}{\omega T_I} + \omega T_D\right)j$$

Intersection with the 0 degree axis:

$$\varphi = 0 \quad \Rightarrow \quad Im = 0 \quad \Rightarrow \quad \frac{-1}{\omega T_I} + \omega T_D = 0 \quad \Rightarrow \quad \frac{1}{\omega T_I} = \omega T_D \quad \Rightarrow$$
$$\omega_m = \sqrt{\frac{1}{T_I \cdot T_D}} = \sqrt{\frac{1}{\frac{T_N}{K} \cdot T_V K}} = \sqrt{\frac{1}{T_N \cdot T_V}}$$













## 2.13 Dead Time

Other name: time delay Example: conveyor belt



Figure 2.41: Conveyor belt as an example of a dead time.

Transfer function:

$$G\left(s\right) = \mathrm{e}^{-sT_{T}}$$

Poles: none

Zeroes: none

Frequency response:

$$G(\mathbf{j}\omega) = \mathbf{e}^{-\mathbf{j}\omega T_T}$$

Amplitude response:

$$A\left(\omega\right) = \left|G\left(\mathrm{j}\omega\right)\right| = 1$$

Phase response:













(b) Block diagram



## 2.14 State Space Representation

Differential equation of a second-order system:

$$\ddot{v} + 2D\omega_0\dot{v} + \omega_0^2 v = K\omega_0^2 u$$

Introduction of 2 state variables:

$$x_1 = v$$
 (Displacement)  
 $x_2 = \dot{v}$  (Velocity)

First differential equation:

 $\dot{x}_1 = x_2$ 

Second differential equation:

$$\dot{x}_2 + 2D\omega_0 x_2 + \omega_0^2 x_1 = K\omega_0^2 u$$
$$\dot{x}_2 = -\omega_0^2 x_1 - 2D\omega_0 x_2 + K\omega_0^2 u$$

Matrix notation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2D\omega_0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ K\omega_0^2 \end{bmatrix}}_{B=b} u$$

Vector differential equation in state space:

$$\dot{m{x}} = m{A} \cdot m{x} + m{B} \cdot m{u}$$

Vector output equation:

$$v = C \cdot x + D \cdot u$$

Output equation in explicit form:

$$v = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C = c^T} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{D = d} u$$

#### 2.14.1 Block Diagram of the State Space Representation

 $\checkmark$  "One integrator per state"

 $\checkmark$  "Model the differential equation at the input of the integrator"



Figure 2.44: Block diagram of a second-order oscillator.

# Chapter 3

# **Controller Design**

## 3.1 Stability

#### 3.1.1 BIBO-Stability (Bounded-Input Bounded-Output)

"Stable when a limited input signal leads to a limited output signal."

#### 3.1.1.1 Example: Step Response of an Integrator



Figure 3.1: Example: Step response of an integrator.

Result: integrator is not stable.

#### 3.1.2 Asymptotic Stability

- stable if the impulse response asymptotically decays to zero.
- unstable if the impulse response approaches infinity
- marginally stable if the impulse response does not exceed a finite value

#### 3.1.2.1 Example: Impulse Response of an Integrator



Figure 3.2: Example: impulse response of an integrator

Result: integrator is marginally stable.

#### 3.1.2.2 Example: Impulse Response of a Double Integrator



Figure 3.3: Example: impulse response of a double integrator.

Result: the double integrator is unstable.

#### 3.1.3 Fundamental Stability Criterion

- stable if the transfer function only has poles in the left half plane
- unstable if at least one pole lies in the right half plane or if at least one multiple pole lies on the imaginary axis
- marginally stable if no pole lies in the right half plane, there are no multiple poles on the imaginary axis, but there is at least one simple pole on the imaginary axis

#### 3.1.3.1 Example: General Second-Order System

$$G(s) = \frac{1}{s^2 + a_1 s + a_0} \quad \Rightarrow \quad s_{1,2} = \dots$$



Figure 3.4: Pole distributions of stable and unstable systems.

## 3.2 Control Loop



Figure 3.5: General control loop.

"From the output backwards to all inputs, or to the output itself":

$$x = G_S \cdot y = G_S \cdot G_R \cdot e = G_S \cdot G_R \cdot (w - r) = G_S \cdot G_R \cdot (w - G_M \cdot x)$$

Sort:

$$x \cdot (1 + G_S \cdot G_R \cdot G_M) = G_S \cdot G_R \cdot w$$

Overall transfer function:

$$G_g = \frac{x}{w} = \frac{G_S \cdot G_R}{1 + G_S \cdot G_R \cdot G_M} = \frac{G_V}{1 + G_0}$$

Forward transfer function:

$$G_V = G_S \cdot G_R$$

Open loop transfer function:

$$G_0 = G_S \cdot G_R \cdot G_M$$

#### 3.2.1 Simple Control Loop Example



Figure 3.6: Example of a simple control loop.

Overall transfer function:

$$G_g = \frac{G_V}{1+G_0} = \frac{K_R \cdot \frac{1}{10s+1}}{1+K_R \cdot \frac{1}{10s+1} \cdot \frac{1}{s+1}} = \frac{K_R \left(s+1\right)}{\left(10s+1\right) \left(s+1\right) + K_R}$$

Poles:

$$10s^2 + 10s + s + 1 + K_R = 0$$

Normal form:

$$s^2 + \frac{11}{10}s + \frac{1+K_R}{10} = 0$$

Two (real or complex conjugate) poles:

$$s_{1,2} = -\frac{11}{20} \pm \sqrt{\left(\frac{11}{20}\right)^2 - \frac{1+K_R}{10}}$$

For the special controller gain  $K_R = 1$ :

$$s_{1,2} = -\frac{11}{20} \pm \sqrt{\left(\frac{11}{20}\right)^2 - \frac{2}{10}}$$

there are two real (stable) poles:

$$s_1 = -0.2$$
 and  $s_2 = -0.87$ 

For the special controller gain  $K_R = 10$ :

$$s_{1,2} = -\frac{11}{20} \pm \sqrt{\left(\frac{11}{20}\right)^2 - \frac{11}{10}}$$

there are two complex conjugate (stable) poles:

$$s_{1,2} = -0.55 \pm 0.89$$
j



Figure 3.7: Poles of a simple control loop depending on the controller gain.

## 3.3 Nyquist Criterion

Preliminary consideration:



Figure 3.8: When does the acoustic loop become unstable?

Feedback only if :

- 1. Gain large enough (> 1) and
- 2. Positive feedback present (phase shift =  $n \cdot 2\pi$ )

Control loop:



Figure 3.9: General control loop.

Open loop transfer function:

 $G_0 = G_R \cdot G_S \cdot G_M$ 

Critical point:

$$A = |G_0| = 1 \quad \text{and} \quad \varphi = \angle G_0 = -\pi$$

## 3.3.1 Example

P-Controller

$$G_R = K_R$$

PT2 plant:

$$G_S = \frac{0.1}{s^2 + s + 1}$$

PT1 sensor:

$$G_M = \frac{1}{0.1s + 1}$$



Figure 3.10: Block diagram of a control loop with P-controller, PT2-plant and PT1-sensor.

Open loop transfer function:

$$G_0 = G_R \cdot G_S \cdot G_M = K_R \cdot \frac{0.1}{s^2 + s + 1} \cdot \frac{1}{0.1s + 1}$$



Figure 3.11: Bode diagram of the open loop transfer function  $(G_0)$ .

Shift up to 0 dB (amplitude reserve):

 $K_{R_{crit}} \approx 40 \,\mathrm{dB} = 100 \quad \text{(closed loop is marginally stable)}$ 



Figure 3.12: Nyquist plot of the open loop transfer function  $(G_0)$ .

Simplified Nyquist criterion on the Nyquist curve of the *open* loop: Critical point (-1) must be on the left, then the *closed* circle  $G_g$  is stable! (Requirement: stable open loop plus a maximum of two integrators)

#### 3.3.2 Two Methods to Examine the Stability of the Control Loop

- 1. calculate the *closed* circle transfer function and apply the **fundamental stability criterion** to the closed circle (poles of the closed circle in the left half plane)
- 2. calculate the *open* loop transfer function  $(G_0 = G_R G_S G_M)$  and apply the **Nyquist** criterion

#### 3.4 Controller Design

Three (sometimes contradictory) demands:

- 1. stability
- 2. speed
- 3. small control error



Figure 3.13: General control loop with disturbance and segmented plant.

Open loop transfer functions:

$$G_0 = G_R \cdot G_{S_1} \cdot G_{S_2}$$

Reference transfer function:

$$G_g = \frac{x}{w} = \frac{G_0}{1+G_0} \quad (\text{preferably} \to 1)$$

Disturbance transfer function:

$$G_z = \frac{x}{z} = \frac{G_{S_2}}{1 + G_0} \quad (\text{preferably} \to 0)$$

Reference control error:

$$G_{ew} = \frac{e}{w} = \frac{1}{1+G_0} \quad (\text{preferably} \to 0)$$

Disturbance control error:

$$G_{ez} = \frac{e}{z} = -\frac{G_{S_2}}{1+G_0} \quad (\text{preferably} \to 0)$$

#### 3.4.1 Example: PT3

$$G_{S_1} = \frac{1}{s+1} \\ G_{S_2} = \frac{2}{s^2 + s + 1}$$

P-controller:

 $G_R = K_R$ 

Open loop transfer function:

$$G_0 = \frac{2K_R}{(s+1)(s^2+s+1)}$$

Closed loop transfer function:

$$G_g = \frac{G_0}{1+G_0} = \frac{\frac{2K_R}{(s+1)(s^2+s+1)}}{1+\frac{2K_R}{(s+1)(s^2+s+1)}} = \frac{2K_R}{(s+1)(s^2+s+1)+2K_R}$$

Good control accuracy:

$$K_R \to \infty \quad \Rightarrow \quad G_g \to 1$$

But: if  $K_R$  is too large  $\Rightarrow$  control loop might become unstable. Steady-state control error transfer function:

$$G_{ew} = \frac{1}{1+G_0} = \frac{1}{1+\frac{2K_R}{(s+1)(s^2+s+1)}} = \frac{(s+1)(s^2+s+1)}{(s+1)(s^2+s+1)+2K_R}$$

Small control error:

$$K_R \to \infty \quad \Rightarrow \quad G_{ew} \to 0$$

But: see above

Limit theorem of the Laplace transform:

$$\lim_{t \to \infty} h\left(t\right) = \lim_{s \to 0} G\left(s\right)$$

Reference step:

 $w = s\left(t\right)$ 

Steady-state control error:

$$\lim_{t \to \infty} e = e_{\infty} = \lim_{s \to 0} G_{ew}$$
$$e_{\infty} = \lim_{s \to 0} \frac{(s+1)(s^2 + s + 1)}{(s+1)(s^2 + s + 1) + 2K_R} = \frac{1}{1 + 2K_R}$$

Increasing the controller gain reduces the steady-state control deviation:

$$K_R \uparrow \Rightarrow e_{\infty} \downarrow$$

When using an I-controller:

$$G_R = \frac{1}{T_I \cdot s}$$

Open loop transfer function:

$$G_0 = \frac{2}{T_I s \left(s+1\right) \left(s^2+s+1\right)}$$

Steady-state control error transfer function:

$$G_{ew} = \frac{1}{1+G_0} = \frac{1}{1+\frac{2}{T_I s(s+1)(s^2+s+1)}} = \frac{T_I s\left(s+1\right)\left(s^2+s+1\right)}{T_I s\left(s+1\right)\left(s^2+s+1\right)+2}$$

Steady-state control error:

$$e_{\infty} = \lim_{s \to 0} G_{ew} = \frac{T_I 0 (0+1) (0^2 + 0 + 1)}{T_I 0 (0+1) (0^2 + 0 + 1) + 2} = \frac{0}{2} = 0$$

No steady-state control error with I-controller (disadvantage: slower, destabilization)

## 3.5 Quality Criteria



Figure 3.14: Reference step response.

 $e_{max}$  maximum overshoot

 $t_{max}$   $e_{max}$  occurs.

- $T_u$  delay time (inflection point tangent  $\cap 0\%$ )
- $T_g$  compensation time (inflection point tangent  $\cap 0 \% \cap 100 \%$ )
- $t_{an}$  response time (curve  $\cap 100\%$ )
- $t_{\varepsilon}$  the magnitude of the control error stays less than  $\varepsilon$  (z. B.  $t_{3\%}$ )



Figure 3.15: Disturbance step response.

Cost function:

$$K = K_1 \cdot t_{an} + K_2 \cdot t_{\varepsilon} + K_3 \cdot e_{max} + K_4 \cdots = \min \lim \downarrow$$

 $\Rightarrow$   $K_i$  arbitrarily chosen  $\Rightarrow$  compromise

#### 3.5.1 Integral Criteria

$$I = \min \downarrow$$

Integral Error (IE):

$$I = \int_{0}^{\infty} e(t) \cdot dt \quad (e > 0)$$

Integral Absolute Error (IAE):

$$I = \int_{0}^{\infty} |e(t)| \cdot dt \quad \text{(inefficient)}$$

Integral Square Error (ISE):

$$I = \int_{0}^{\infty} e^{2}(t) \cdot dt \quad \text{(analytical calculation)}$$

Integral Time Square Error (ITSE):

$$I = \int_{0}^{\infty} e^{2}(t) \cdot t \cdot dt \quad \text{(duration of the control error)}$$

Control effort:

$$I = \int_{0}^{\infty} \left( e^{2}(t) + \alpha \cdot y^{2}(t) \right) \cdot dt \quad (\alpha: \text{ subjective weighting factor})$$

## 3.6 Controller Optimisation with Simulation



Figure 3.16: Controller optimisation.

## 3.7 Control Loop Tuning Rules

PID-controller (compare section 2.12)

P basic control

I steady-state accuracy (slow)

D speed (I compensation)

Two proven methods:

#### 3.7.1 Ziegler and Nichols (Margin of Stability)

Turn up the P-controller until the control loop oscillates stationary:

 $\rightarrow K_{R_{krit}}$ 

Measure the period of the oscillation:

 $\rightarrow T_{krit}$ 

Controller	Controller gain $K_R$	Reset time $T_N$	Rate time $T_V$
Р	$0.5 K_{R_{krit}}$	-	-
PI	$0.45 K_{R_{krit}}$	$0.85T_{krit}$	-
PID	$0.6 K_{R_{krit}}$	$0.5T_{krit}$	$0.12  T_{krit}$

Table 3.1: Controller Gain, Reset Time, and Rate Time Dependence on Oscillation Test Parameters

## 3.7.2 Chien, Hrones, and Reswick (Step Response)



Figure 3.17: Step response of the plant.

- inflection point tangent  $\rightarrow$  delay time  $T_u$ , compensation time  $T_g$
- applicable if  $T_g/T_u > 3$

Definition of an auxiliary variable:

$$K_H = \frac{T_g}{K_S T_u}$$

Controller	Optimized for	Overshoot	$K_R$	$T_N$	$T_V$
Р	disturbance	0 %	$0.3 K_H$	-	-
		20%	$0.7 K_H$	-	-
	reference	0 %	$0.3 K_H$	-	-
		20%	$0.7 K_H$	-	-
PI	disturbance	0 %	$0.6 K_H$	$4.0 T_u$	-
		20%	$0.7 K_H$	$2.3 T_u$	-
	reference	0%	$0.35 K_H$	$1.2 T_g$	-
		20%	$0.6 K_H$	$1.0 T_g$	-
PID	disturbance	0 %	$0.95 K_H$	$2.4 T_u$	$0.42 T_u$
		20%	$1.2 K_H$	$2.0 T_u$	$0.42 T_u$
	reference	0 %	$0.6 K_H$	$1.0 T_{g}$	$0.5 T_u$
		20%	$0.95 K_H$	$1.35 T_g$	$0.47 T_u$

Table 3.2: Controller Gain, Reset Time, and Rate Time Dependence on Steady-State Amplitude, Delay Time, and Compensation Time

## 3.8 Disturbance Feedforward



Figure 3.18: Disturbance feedforward.

Concept: compensation for the disturbance variable z if it can be measured. If possible, the disturbance variable should have no influence on the output variable x:

Transfer function from z to x:

$$G_{xz} = \frac{G_z + G_{St}G_S}{1 + G_R G_S} \stackrel{!}{=} 0$$

Numerator of  $G_{xz}$  must disappear:

$$G_z + G_{St}G_S = 0$$

Condition for disturbance feedforward transfer function:

$$G_{St} = -\frac{G_z}{G_S}$$

Problem:  $G_{St}$  cannot always be implemented exactly  $\Rightarrow$  at least stationary disturbance compensation (s = 0) Alternative: signal injection before the controller: New transfer function from z to x:

$$G_{xz} = \frac{G_z + G'_{St}G_RG_S}{1 + G_RG_S} \stackrel{!}{=} 0$$

Numerator must be zero:

$$G_z + G'_{St}G_RG_S = 0$$

Compensation condition:

$$G'_{St} = -\frac{G_z}{G_R G_S}$$

Advantage: no actuator energy necessary Disadvantage: signal has to go through the controller

## 3.9 Open Loop Control (Feedforward)



Figure 3.19: Pure open loop control.

Requirement:

Ideal:

$$G_V = \frac{1}{G_S}$$

x = w

e.g.:

$$G_S = \frac{2}{3s+1} \quad \Rightarrow \quad G_V = \frac{3s+1}{2}$$

Problems:

- 1.  $G_S$  is not exactly known.
- 2.  $G_S$  is not exactly invertible (dead time, pure integrator, ...).
- 3. disturbances are not detected.

Steady-state open loop control  $G_{V_{st}}$  meets the requirement x = w at least after the transient response  $(t \to \infty \implies s = 0)$ :

Example above:

$$G_{V_{st}} = \lim_{s \to 0} \frac{3s+1}{2} = \frac{3 \cdot 0 + 1}{2} = 0.5$$

#### 3.9.1 Combination with Closed Loop Control



Figure 3.20: Combination of open loop and closed loop control.

#### Method:

- 1. design the open loop controller  $G_V$  for best possible inversion.
- 2. design the closed loop controller  $G_R$  (just has to compensate the inadequacies of the open loop controller).

## 3.10 Digital (Discrete-Time) Control



Figure 3.21: Digital control (T: sampling time).



Figure 3.22: Sampling with digital control.

#### 3.10.1 z-Transformation

The transfer function in the s-domain is given:

$$G\left(s\right) = \frac{V\left(s\right)}{U\left(s\right)}$$

We are looking for the transfer function in the z-domain:

$$G\left(z\right) = \frac{v\left(kT\right)}{u\left(kT\right)}$$

and derived from this, the difference equation:

$$v_{k+1} = f(v_k, v_{k-1}, \dots, u_{k+1}, u_k, u_{k-1}, \dots)$$

## 3.10.2 Approximations

Rectangle rule:

$$s \approx \frac{z-1}{T \cdot z}$$

Tustin's rule:

$$s \approx \frac{2}{T} \cdot \frac{z-1}{z+1}$$

#### 3.10.3 Example: Digital Low Pass

Transfer function in the s-domain:

$$G\left(s\right) = \frac{2}{5s+1}$$

Sampling time:

$$T = 0.1$$

Using Tustin's rule:

$$s\approx 20\cdot\frac{z-1}{z+1}$$

Transfer function in the z-domain:

$$G\left(z\right) = \frac{2}{5 \cdot 20 \cdot \frac{z-1}{z+1} + 1} = \frac{2\left(z+1\right)}{100\left(z-1\right) + z + 1} = \frac{2z+2}{101z-99}$$

Quotient of output and input variable:

$$G(z) = \frac{v_k}{u_k} = \frac{2z+2}{101z-99}$$

Multiply crosswise:

$$v_k \left( 101z - 99 \right) = u_k \left( 2z + 2 \right)$$

"Multiplication by z means shifting in the positive time direction":

 $v_k \cdot z \stackrel{\circ}{=} v_{k+1}$ 

Difference equation:

$$101v_{k+1} - 99v_k = 2u_{k+1} + 2u_k$$

Solve for "new output variable":



Figure 3.23: Discrete implementation of a digital low pass filter.

# Part II Flight Control

# Chapter 4

# Introduction

## 4.1 Aviation Terminology

- All symbols are consistent with European standards [1] [2] (with extensions:  $V_A$ ,  $\Omega_K$ )
- All axis systems are right-handed (i.e., use the 'right hand rule').
- A summary of the nomenclature and self-test is available [3]
- Many illustrations are reproduced (with the kind permission of the author) from [4].

#### 4.1.1 Variables of Motion

There are define several different orthogonal axis systems for use in aerospace. Figure 4.1 indicates positive directions and moments with axes centered on an aircraft.



Figure 4.1: Positive directions and moments according to [1].
Position vector (direction, distance):

$$\boldsymbol{s} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \begin{array}{c} \text{towards the front} \\ \text{towards the right} \\ \text{towards the bottom} \end{array}$$

Force vector:

$$\boldsymbol{R} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Moment vector:

$$oldsymbol{Q} = \begin{bmatrix} L \\ M \\ N \end{bmatrix}$$

Velocity vector:

$$oldsymbol{V} = egin{bmatrix} u \ v \ w \end{bmatrix}$$

Angular velocity vector:

$$\boldsymbol{\Omega} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \quad \begin{array}{c} \text{roll} \\ \text{pitch} \\ \text{yaw} \end{array}$$

Attitude vector (Euler angle vector):

$$oldsymbol{\Phi} = egin{bmatrix} \Phi \ \Theta \ \Psi \end{bmatrix}$$

# 4.1.2 Indices

The following indices are used to denote different types of force, moment or velocity:

- A aerodynamic
- K flight-path
- W wind
- F thrust

Lower case subscripts are used to denote different coordinate systems:

- f body-fixed axis system
- *a* aerodynamic axis system
- k flight-path axis system
- g Earth-fixed (geodetic) axis system

### 4.1.2.1 Axis System Examples

Some examples of forces and moments in different coordinate systems:

Resulting aerodynamic force in the aerodynamic axis system:  $R_a^A$ 

Resultant aerodynamic moment in the flight-path axis system:  $oldsymbol{Q}_k^A$ 

Thrust force in the body-fixed axis system:  $F_f^F$ 

Thrust moment:  $Q^F$ 

Weight force in the geodetic axis system:  $G_g$ 

Gravitational acceleration vector (acceleration due to gravity) in the body-fixed axis system:  $\boldsymbol{g_f}$ 

# 4.1.3 Velocities



Figure 4.2: Aerodynamic velocity, flight-path velocity, and wind velocity.

Flight-path velocity  $V_K$ Relative velocity of the aircraft with respect to the EarthAerodynamic velocity  $V_A$ Relative velocity of the aircraft with respect to the airWind velocity  $V_W$ Relative velocity of the air with respect to the Earth

Relationship between flight-path velocity, aerodynamic velocity and wind velocity vectors:

$$V_K = V_A + V_W$$

Equivalent relationship for angular velocity vectors:

$$arOmega_K = arOmega_A + arOmega_W$$

### 4.1.4 Control Variables



Figure 4.3: A positive stick deflection (pull) leads to a negative elevator deflection and thus to a positive pitch moment.



Figure 4.4: A positive stick deflection (to the right) leads to a negative aileron deflection (right aileron up) and thus to a positive roll moment.



Figure 4.5: A negative pedal angle (right pedal depressed) leads to a negative rudder deflection and thus to a positive yaw moment.

# 4.2 Coordinate Transformation

The general three-dimensional vector can be described in any axis system.

$$oldsymbol{V} = egin{bmatrix} u \ v \ w \end{bmatrix}$$

For example:



(a) Vector  $\boldsymbol{V}$ , represented in the *a* axis system



(b) Same vector  $\boldsymbol{V}$ , represented in the b axis system



Vector  $\boldsymbol{V}$ , expressed in the *a* axis system:

$$\boldsymbol{V_a} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}_a = \begin{bmatrix} u_a \\ v_a \\ w_a \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The *b* axis system is created by a 90° rotation of the *a* axis system about the  $y_a$  axis. The vector V is not rotated in the process.

Same vector  $\boldsymbol{V}$ , expressed in the *b* axis system:

$$\boldsymbol{V_b} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}_b = \begin{bmatrix} u_b \\ v_b \\ w_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The vector (expressed in the new axis system) now has different coordinates; however, it is still the same vector.

It is often useful to express aerodynamic quantities in different axis systems, to simplify analysis or communication.

### 4.2.1 Axis Systems

Figure 4.7 is a schematic of the various axis systems used in aviation, and the transformations that relate them to each other.



Figure 4.7: Aviation axis systems (coordinate systems) and transformation matrices (according to [1]).

• The geodetic (Earth-fixed) axis system (index: g) is defined by its  $z_g$  axis pointing in the direction of gravity. The  $x_g$  axis is perpendicular to the  $z_g$  axis in the Earth's horizontal plane and is often assumed to be in the north direction. The  $y_g$  axis forms (as in all described axis systems) a right-handed axis system with the other two axes and therefore also lies in the Earth's horizontal plane.

- The body-fixed axis system (index: f or no index) describes the attitude of the aircraft in space. The  $x_f$  axis points "forwards" (usually in the plane of symmetry from the centre of gravity to the nose of the aircraft), the  $y_f$  axis points "to the right" (starboard) and the  $z_f$  axis points "downwards".
- The aerodynamic axis system (index: a) is defined by its  $x_a$  axis, which points in the direction of aerodynamic velocity vector  $V_A$ . Since the axis system is not yet clearly defined by the definition of one axis (it could still rotate about its  $x_a$  axis), the  $z_a$  axis is defined in the plane of aircraft symmetry ( $x_f$ - $z_f$  plane). This means that the  $y_f$  axis also lies in the  $x_a$ - $y_a$  plane (cf. figure ??).
- The flight-path axis system (index: k) is defined analogously to the aerodynamic axis system: The  $x_k$  axis points in the direction of the flight-path velocity vector  $V_K$ . As a second definition, the  $y_k$  axis is usually placed in the Earth's horizontal plane ( $x_g$ - $y_g$  plane) (cf. figure 4.9).

### 4.2.2 Rotation from Geodetic to Body-fixed Axes

Figure 4.8 shows the relationship between the geodetic and body-fixed coordinate systems.



Figure 4.8: Euler angle rotation from the geodetic to the body-fixed axis system [5] (according to [1]).

• The yaw angle (heading, azimuth angle)  $\Psi$  rotates in the  $x_g$ - $y_g$  plane about the  $z_g$  axis. In doing so, the  $x_g$  axis is transformed into the nodal axis  $k_1$  and the  $y_g$  axis into the nodal axis  $k_2$ . Main value range::  $-\pi < \Psi \leq \pi$ 

- The pitch angle (inclination angle)  $\Theta$  rotates in the  $x_f$ - $z_g$  plane about the  $k_2$  axis. In the process, the  $k_1$  axis is transformed into the  $x_f$  axis and the  $z_g$  axis into the nodal axis  $k_3$ . Main value range:  $-\frac{\pi}{2} \leq \Theta \leq \frac{\pi}{2}$
- The roll angle (bank angle)  $\Phi$  rotates in the  $y_f$ - $z_f$  plane about the  $x_f$  axis. In doing so, the  $k_2$  axis is transformed into the  $y_f$  axis and the  $k_3$  axis into the  $z_f$  axis. Main value range:  $-\pi < \Phi \leq \pi$

# 4.2.3 Rotation from the geodetic to the flight-path axis system



Figure 4.9: Rotation from the geodetic to the flight-path axis system [5].

- The flight-path azimuth angle  $\chi$  rotates in the  $x_g$ - $y_g$  plane about the  $z_g$  axis. In doing so, the  $x_g$  axis is transformed into the nodal axis  $k_k$  and the  $y_g$  axis into the  $y_k$  axis. Main value range:  $-\pi < \chi \leq \pi$
- The flight-path angle (angle of climb, flight-path inclination angle)  $\gamma$  rotates in the  $x_k$ - $z_k$  plane about the  $y_k$  axis. In the process, the  $k_k$  axis is transformed into the  $x_k$  axis and the  $z_g$  axis into the  $z_k$  axis. Main value range:  $-\frac{\pi}{2} \leq \gamma \leq \frac{\pi}{2}$
- A rotation about the  $x_k$  axis (as with the Euler angles with  $\Phi$ ) does not take place, since the  $y_k$  axis lies in the Earth's horizontal plane ( $x_g$ - $y_g$  plane) by definition.

4.2.4 Rotation from the aerodynamic to the body-fixed axis system



Figure 4.10: Rotation from the aerodynamic to the body-fixed axis system [5]

- The side-slip angle  $\beta$  rotates in the  $x_a$ - $y_a$  plane about the  $z_a$  axis. By rotating in a negative direction, i. e. by "minus beta", the  $x_a$  axis is transformed into the intermediate  $x_e$  axis (experimental axis system) and the  $y_a$  axis into the  $y_f$  axis. Main value range:  $-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}$
- The angle of attack  $\alpha$  rotates in the  $x_f z_f$  plane about the  $y_f$  axis. In the process, the  $x_e$  axis is transformed into the  $x_f$  axis and the  $z_a$  axis into the  $z_f$  axis. Main value range:  $-\pi < \alpha \leq \pi$
- A rotation about the  $x_f$  axis (as with the Euler angles with  $\Phi$ ) does not take place, since the  $z_a$  axis lies in the plane of aircraft symmetry ( $x_f$ - $z_f$  plane) by definition.

### 4.2.5 Transformation matrices

Rotation with angle  $w_z$  about a z axis:

$$\boldsymbol{M_z} = \begin{bmatrix} \cos w_z & \sin w_z & 0\\ -\sin w_z & \cos w_z & 0\\ 0 & 0 & 1 \end{bmatrix}$$

Rotation with angle  $w_y$  about a y axis:

$$\boldsymbol{M_y} = \begin{bmatrix} \cos w_y & 0 & -\sin w_y \\ 0 & 1 & 0 \\ \sin w_y & 0 & \cos w_y \end{bmatrix}$$

Rotation with angle  $w_x$  about an x axis:

$$\boldsymbol{M}_{\boldsymbol{x}} = \begin{bmatrix} 1 & 0 & 0\\ 0 & \cos w_x & \sin w_x\\ 0 & -\sin w_x & \cos w_x \end{bmatrix}$$

Total transformation matrix with a rotation sequence  $w_z \to w_y \to w_x$  ("read from the right"):

$$\boldsymbol{M_{ges}} = \boldsymbol{M_x} \cdot \boldsymbol{M_y} \cdot \boldsymbol{M_z}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos w_x & \sin w_x \\ 0 & -\sin w_x & \cos w_x \end{bmatrix} \begin{bmatrix} \cos w_y & 0 & -\sin w_y \\ 0 & 1 & 0 \\ \sin w_y & 0 & \cos w_y \end{bmatrix} \begin{bmatrix} \cos w_z & \sin w_z & 0 \\ -\sin w_z & \cos w_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Transformation from the geodetic to the body-fixed axis system:

$$\mathbf{M}_{fg} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \Phi & \sin \Phi \\ 0 & -\sin \Phi & \cos \Phi \end{bmatrix} \begin{bmatrix} \cos \Theta & 0 & -\sin \Theta \\ 0 & 1 & 0 \\ \sin \Theta & 0 & \cos \Theta \end{bmatrix} \begin{bmatrix} \cos \Psi & \sin \Psi & 0 \\ -\sin \Psi & \cos \Psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos \Theta \cos \Psi & \cos \Theta \sin \Psi & -\sin \Theta \\ \sin \Phi \sin \Theta \cos \Psi - \cos \Phi \sin \Psi & \sin \Phi \sin \Theta \sin \Psi + \cos \Phi \cos \Psi & \sin \Phi \cos \Theta \\ \cos \Phi \sin \Theta \cos \Psi + \sin \Phi \sin \Psi & \cos \Phi \sin \Theta \sin \Psi - \sin \Phi \cos \Psi & \cos \Phi \cos \Theta \end{bmatrix}$$
(4.1)

Transformation from the aerodynamic to the body-fixed axis system:

$$\begin{split} \boldsymbol{M_{fa}} &= \begin{bmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos(-\beta) & \sin(-\beta) & 0 \\ -\sin(-\beta) & \cos(-\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \beta & -\sin \beta & 0 \\ \sin \beta & \cos \beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos \alpha \cos \beta & -\cos \alpha \sin \beta & -\sin \alpha \\ \sin \beta & \cos \beta & 0 \\ \sin \alpha \cos \beta & -\sin \alpha \sin \beta & \cos \alpha \end{bmatrix} \end{split}$$

Transformation from the geodetic to the flight-path axis system:

$$\boldsymbol{M_{kg}} = \begin{bmatrix} \cos\gamma & 0 & -\sin\gamma \\ 0 & 1 & 0 \\ \sin\gamma & 0 & \cos\gamma \end{bmatrix} \begin{bmatrix} \cos\chi & \sin\chi & 0 \\ -\sin\chi & \cos\chi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \cos\gamma\cos\chi & \cos\gamma\sin\chi & -\sin\gamma \\ -\sin\chi & \cos\chi & \sin\gamma\sin\chi & -\sin\gamma \\ \sin\gamma\cos\chi & \sin\gamma\sin\chi & \cos\gamma \end{bmatrix}$$

#### 4.2.5.1 Transformation direction reversal

Two ways to generate the inverse transformation (reverse transformation):

1. by reversing the order of the single transformations and negative angles:

$$\begin{split} \boldsymbol{M_{gk}} &= \begin{bmatrix} \cos(-\chi) & \sin(-\chi) & 0\\ -\sin(-\chi) & \cos(-\chi) & 0\\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(-\gamma) & 0 & -\sin(-\gamma)\\ 0 & 1 & 0\\ \sin(-\gamma) & 0 & \cos(-\gamma) \end{bmatrix} \\ &= \begin{bmatrix} \cos\chi & -\sin\chi & 0\\ \sin\chi & \cos\chi & 0\\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\gamma & 0 & \sin\gamma\\ 0 & 1 & 0\\ -\sin\gamma & 0 & \cos\gamma \end{bmatrix} \\ &= \begin{bmatrix} \cos\chi\cos\gamma & -\sin\chi & \cos\chi\sin\gamma\\ \sin\chi\cos\gamma & \cos\chi & \sin\chi\sin\gamma\\ -\sin\gamma & 0 & \cos\gamma \end{bmatrix} \end{split}$$

2. by inverting the transformation matrix. With the rotational transformations used, inverting is simplified to transposing:

$$M_{gk} = M_{kg}^{-1} = M_{kg}^{T}$$
$$= \begin{bmatrix} \cos\gamma\cos\chi & \cos\gamma\sin\chi & -\sin\gamma\\ -\sin\chi & \cos\chi & 0\\ \sin\gamma\cos\chi & \sin\gamma\sin\chi & \cos\gamma \end{bmatrix}^{T}$$
$$= \begin{bmatrix} \cos\gamma\cos\chi & -\sin\chi & \sin\gamma\cos\chi\\ \cos\gamma\sin\chi & \cos\chi & \sin\gamma\sin\chi\\ -\sin\gamma & 0 & \cos\gamma \end{bmatrix}$$

#### 4.2.5.2 Coordinate Transform Example

The weight vector has only a z component in the geodetic axis system, namely its magnitude:

$$\boldsymbol{G_g} = \begin{bmatrix} 0\\ 0\\ mg \end{bmatrix}$$

After the transformation into the body-fixed axis system, the weight vector is dense (fully occupied):

$$\begin{aligned} \boldsymbol{G}_{f} &= \boldsymbol{M}_{fg} \boldsymbol{G}_{g} = \begin{bmatrix} \cdots & \cdots & -\sin \Theta \\ \cdots & \cdots & \sin \Phi \cos \Theta \\ \cdots & \cdots & \cos \Phi \cos \Theta \end{bmatrix} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ mg \end{bmatrix} \\ &= \begin{bmatrix} -\sin \Theta \cdot mg \\ \sin \Phi \cos \Theta \cdot mg \\ \cos \Phi \cos \Theta \cdot mg \end{bmatrix} = \begin{bmatrix} -\sin \Theta \\ \sin \Phi \cos \Theta \\ \cos \Phi \cos \Theta \end{bmatrix} \cdot mg \end{aligned}$$

## 4.2.6 Conversion between Cartesian and Spherical Coordinates

#### 4.2.6.1 Aerodynamic Velocity Conversion

The airspeed vector  $V_A$  can be expressed particularly simply in the aerodynamic axis system due to its definition. It has only a  $u_A$  component there:

$$\boldsymbol{V_{Aa}} = \begin{bmatrix} u_A \\ v_A \\ w_A \end{bmatrix}_a = \begin{bmatrix} V_A \\ 0 \\ 0 \end{bmatrix}$$

After the transformation into the body-fixed axis system, the relations between the Cartesian and the spherical coordinates of the aerodynamic velocity vector are obtained:

$$\boldsymbol{V_{Af}} = \begin{bmatrix} u_A \\ v_A \\ w_A \end{bmatrix}_f = \boldsymbol{M_{fa}}\boldsymbol{V_{Aa}} = \begin{bmatrix} \cos\alpha\cos\beta & \cdots & \cdots \\ \sin\beta & \cdots & \cdots \\ \sin\alpha\cos\beta & \cdots & \cdots \end{bmatrix} \begin{bmatrix} V_A \\ 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} V_A\cos\alpha\cos\beta \\ V_A\sin\beta \\ V_A\sin\beta \\ V_A\sin\alpha\cos\beta \end{bmatrix}}_{\text{spherical}}$$

Check the identity of the norms of both representations:

$$\begin{aligned} |\mathbf{V}_{\mathbf{A}}| &= \sqrt{u_{Af}^2 + v_{Af}^2 + w_{Af}^2} \\ &= \sqrt{V_A^2 \cos^2 \alpha \cos^2 \beta + V_A^2 \sin^2 \beta + V_A^2 \sin^2 \alpha \cos^2 \beta} \\ &= \sqrt{V_A^2 \underbrace{\left(\cos^2 \alpha + \sin^2 \alpha\right)}_{1} \cos^2 \beta + V_A^2 \sin^2 \beta} \\ &= \sqrt{V_A^2 \underbrace{\left(\cos^2 \beta + \sin^2 \beta\right)}_{1}} = V_A \quad \text{q. e. d} \end{aligned}$$

The ratio of the two Cartesian coordinates is:

$$\frac{w_{Af}}{u_{Af}} = \frac{V_A \sin \alpha \cos \beta}{V_A \cos \alpha \cos \beta} = \tan \alpha$$

Therefore, the solution for the angle of attack becomes:

$$\alpha = \arctan\left(\frac{w_{Af}}{u_{Af}}\right)$$

Second Cartesian coordinate:

$$v_{Af} = V_A \cdot \sin \beta$$

The solution for the side-slip angle is:

$$\beta = \arcsin\left(\frac{v_{Af}}{V_A}\right)$$

#### 4.2.6.2 Conversion of the Flight-Path Velocity

The flight-path velocity vector  $V_K$  can be expressed particularly simply in the flight-path axis system due to its definition. It has only a  $u_K$  component there:

$$\boldsymbol{V_{Kk}} = \begin{bmatrix} u_K \\ v_K \\ w_K \end{bmatrix}_k = \begin{bmatrix} V_K \\ 0 \\ 0 \end{bmatrix}$$

After the transformation into the geodetic axis system, the relationships between the Cartesian and the spherical coordinates of the flight-path velocity vector are obtained:

$$\boldsymbol{V_{Kg}} = \underbrace{\begin{bmatrix} u_K \\ v_K \\ w_K \end{bmatrix}_g}_{\text{Cartesian}} = \boldsymbol{M_{gk}}\boldsymbol{V_{Kk}} = \begin{bmatrix} \cos\gamma\cos\chi & \cdots & \cdots \\ \cos\gamma\sin\chi & \cdots & \cdots \\ -\sin\gamma & \cdots & \cdots \end{bmatrix} \begin{bmatrix} V_K \\ 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} V_K\cos\gamma\cos\chi \\ V_K\cos\gamma\sin\chi \\ -V_K\sin\gamma \end{bmatrix}}_{\text{spherical}}$$

Ratio of two Cartesian coordinates:

$$\frac{v_{Kg}}{u_{Kg}} = \frac{V_K \cos \gamma \sin \chi}{V_K \cos \gamma \cos \chi} = \tan \chi$$

Solution for the flight-path azimuth:

$$\chi = \arctan\left(\frac{v_{Kg}}{u_{Kg}}\right)$$

Third Cartesian coordinate:

$$w_{Kg} = -V_K \sin \gamma$$

Solution for the angle of climb:

$$\gamma = -\arcsin\left(\frac{w_{Kg}}{V_K}\right)$$

# 4.2.6.3 Summary of the Conversions

Spherical  $\rightarrow$  Cartesian:

$$u_{Af} = V_A \cos \alpha \cos \beta$$
$$v_{Af} = V_A \sin \beta$$
$$w_{Af} = V_A \sin \alpha \cos \beta$$
$$u_{Kg} = V_K \cos \gamma \cos \chi$$
$$v_{Kg} = V_K \cos \gamma \sin \chi$$
$$w_{Kg} = -V_K \sin \gamma$$

Cartesian  $\rightarrow$  spherical:

$$V_A = \sqrt{u_{Af}^2 + v_{Af}^2 + w_{Af}^2}$$
$$\alpha = \arctan\left(\frac{w_{Af}}{u_{Af}}\right)$$
$$\beta = \arcsin\left(\frac{v_{Af}}{V_A}\right)$$
$$V_K = \sqrt{u_{Kg}^2 + v_{Kg}^2 + w_{Kg}^2}$$
$$\gamma = -\arcsin\left(\frac{w_{Kg}}{V_K}\right)$$
$$\chi = \arctan\left(\frac{v_{Kg}}{u_{Kg}}\right)$$

# 4.2.7 Representation of Angles and Vectors



Figure 4.11: Angles and vectors in the x-z plane.



Figure 4.12: Angles and vectors in the x-y plane



Figure 4.13: Angles and vectors in the y-z plane

# Chapter 5

# **Subsystems**

# 5.1 Aerodynamics

The relative velocity  $V_A$  between the aircraft and the air with density  $\rho$  creates dynamic pressure:

$$\bar{q} = \frac{\rho}{2} V_A^2$$

The product of dynamic pressure  $\bar{q}$  and reference wing area S is called the aerodynamic force unit E:

$$E = \bar{q} \cdot S$$

The aerodynamic forces are the product of the aerodynamic force unit with the dimensionless coefficients in each direction:

 $A = E \cdot C_A$ 

Lift:

Drag:

 $W = E \cdot C_W$ 

Side force:

For the moments, a reference length is also required for dimensional reasons. Usually, the mean aerodynamic chord 
$$l_{\mu}$$
 is used today for all moments:

 $Q = E \cdot C_O$ 

Roll moment:

L	=	E	•	$l_{\mu}$	·	$C_l$
L	=	E	•	$l_{\mu}$	•	

Pitch moment:

 $M = E \cdot l_{\mu} \cdot C_m$ 

Yaw moment:

$$N = E \cdot l_{\mu} \cdot C_n$$

# 5.1.1 Coefficients

The coefficients are non-linear functions of the respective aerodynamic parameters.

#### 5.1.1.1 Coefficients of the Longitudinal Motion

Lift coefficient:

$$C_A = C_A (\alpha, \eta, Ma, q, \dot{\alpha}, \ldots)$$

Drag coefficient:

$$C_W = C_W(\alpha, \eta, Ma, \ldots)$$

Pitch moment coefficient:

$$C_m = C_m \left( \alpha, \, \eta, \, Ma, \, q, \, \dot{\alpha}, \, \ldots \right)$$

Alternative modelling of the drag coefficient via the drag polar:



Figure 5.1: Drag polar.

Drag coefficient:

$$C_W = C_{W0} + k \cdot C_A^2,$$

where  $C_{W0}$  is the zero drag (no lift) and  $k \cdot C_A^2$  is the lift induced drag.

#### 5.1.1.2 Coefficients of Lateral Motion

Side force coefficient:

$$C_Q = C_Q \left(\beta, \, p, \, r, \, \xi, \, \zeta, \, \ldots\right)$$

Roll moment coefficient:

$$C_l = C_l \left(\beta, \, p, \, r, \, \xi, \, \zeta, \, \ldots\right)$$

Yaw moment coefficient:

$$C_n = C_n \left(\beta, \, p, \, r, \, \xi, \, \zeta, \, \ldots\right)$$

### 5.1.2 Linear Aerodynamic Derivatives

An aerodynamic derivative is the partial derivative of an aerodynamic coefficient with respect to an aerodynamic influence quantity.

#### 5.1.2.1 Example: Lift Characteristic



Figure 5.2: Lift characteristic (lift coefficient over angle of attack).

Within a working range (in the vicinity of a working point) a linear dependence of the lift coefficient on the angle of attack is assumed. There, the slope of the characteristic curve is constant and corresponds to the derivative  $C_{A\alpha}$ :

Lift due to angle of attack:

$$C_{A\alpha} = \frac{\partial C_A}{\partial \alpha}$$

Analogously, further lift derivatives are defined:

Lift due to elevator deflection:

$$C_{A\eta} = \frac{\partial C_A}{\partial \eta}$$

Lift due to the Mach number::

$$C_{AMa} = \frac{\partial C_A}{\partial Ma}$$

Since the derivatives (just like the coefficients) are dimensionless, an angular speed (unit:  $s^{-1}$ ) must first be rendered dimensionless (normalised) with a reference time constant before it can be partially derived. Usually, the time constant:

$$T_N = \frac{l_\mu}{V_A}$$

is used for normalisation:

Normalised aerodynamic pitch speed:

$$q_A^* = T_N \cdot q_A = \frac{l_\mu}{V_A} \cdot q_A$$

The corresponding derivative is obtained by partial derivation with respect to the normalised angular speed:

Lift due to pitch speed:

$$C_{Aq} = \frac{\partial C_A}{\partial \left(q_A^*\right)}$$

The total lift coefficient, within the framework of the described linear derivative aerodynamics, is composed of the linear combination of the individual influences: Total lift coefficient:

$$C_A = C_{A_0} + C_{A\alpha} \cdot \alpha + C_{A\eta} \cdot \eta + C_{AMa} \cdot Ma + C_{Aq} \cdot q_A^* + \dots$$

The same applies to the other force and moment coefficients:

Total pitch moment coefficient:

$$C_m = C_{m_0} + C_{m\alpha} \cdot \alpha + C_{m\eta} \cdot \eta + C_{mMa} \cdot Ma + C_{mq} \cdot q_A^* + \dots$$

Total side force coefficient:

$$C_Q = C_{Q\beta} \cdot \beta + C_{Qp} \cdot p_A^* + C_{Qr} \cdot r_A^* + C_{Q\xi} \cdot \xi + C_{Q\zeta} \cdot \zeta + \dots$$

Total roll moment coefficient:

$$C_l = C_{l\beta} \cdot \beta + C_{lp} \cdot p_A^* + C_{lr} \cdot r_A^* + C_{l\xi} \cdot \xi + C_{l\zeta} \cdot \zeta + \dots$$

Total yaw moment coefficient:

$$C_n = C_{n\beta} \cdot \beta + C_{np} \cdot p_A^* + C_{nr} \cdot r_A^* + C_{n\xi} \cdot \xi + C_{n\zeta} \cdot \zeta + \dots$$

The individual derivatives are usually designated according to their cause-effect relationship:

Pitch damping (damping of the pitch motion):

 $C_{mq}$ 

Yaw damping (damping of the yaw motion):

 $C_{nr}$ 

Wind vane stability (alignment "into the wind"):

 $C_{n\beta}$ 

Side-slip roll moment (roll moment due to side-slip):

 $C_{l\beta}$ 

Yaw side force ("Side force due to yaw"):

 $C_{Qr}$ 

etc.

# 5.2 Engine

The thrust vector (maximum thrust) is dependent on

- Air inlet and outlet velocity vectors (thrust vector angle, angle of attack, side-slip angle)
- Air density (altitude)
- Mach number
- ...

Low pass behaviour:

$$T_F \cdot \dot{F} + F = F_c$$

with

- $T_F$  engine time constant
- F thrust
- $F_c$  reference thrust

The thrust moment vector is given by:

$$oldsymbol{Q}_{F} = oldsymbol{r}_{F} imes oldsymbol{F}_{F} = egin{bmatrix} r_{x} \ r_{y} \ r_{z} \end{bmatrix} imes egin{bmatrix} F_{x} \ F_{y} \ F_{z} \end{bmatrix}$$

with

 $F_F$  thrust vector

 $r_F$  thrust vector application point (distance of the engine from the reference point)

 $Q_F$  thrust moment vector

# 5.3 Actuator Dynamics

# 5.3.1 Bathtub Experiment

Suppose there was a board halfway up your bathtub with a plastic duck floating on the water underneath.



(a) Drain closed, inlet opened, water level rises, duck rises.



(b) Drain opened, inlet closed, water level drops, duck does not sink.

Figure 5.3: Bathtub experiment.

- As long as the inlet is opened, the outlet is closed, and the duck is not yet touching the board, the duck will rise together with the rising water level.
- As soon as the duck hits the board, it stops at a constant height. The water level continues to rise regardless.
- When the inlet is closed and the outlet is opened, the water level begins to sink. The duck, however, does not sink yet.
- Only when the sinking water level reaches the duck under the board can it sink together with the water.

# 5.3.2 Generalisation

Bathtub	energy storage, integrator, dynamic system	
Inlet or outlet	input variable $u$ of the integrator	
Water level	energy content, state, output variable $v$ of the integrator	
Board	saturation $v_b$ of the output variable	



Figure 5.4: Subsequently limited integrator.

- **Problem:** If the output variable of a dynamic system is limited, it can happen that the internal state variables "run full" and the reaction of the system only becomes visible after an undesired time delay  $T_T$ , although the limited output signal should actually react immediately.
- **Solution:** Additionally stop the corresponding state variables when the output variable runs into its saturation. In the example: Stop the integrator by explicitly setting its input variable to zero: Close the inlet.

Notice:

Never thoughtlessly limit the output of a dynamic system.

# 5.4 Wind

The total wind can be composed of three parts (stationary wind, urbulence (gusts) and wind shear):

$$V_W = V_{W\,Stat} + V_{W\,Turb} + V_{W\,Shear}$$

Wind shear is particularly relevant during the landing phase of an aircraft:



Figure 5.5: Loss of a passenger aircraft at J.F. Kennedy Airport on 24.6.1975.

# 5.4.1 Turbulence



Figure 5.6: White noise: equal power density for all frequencies. Pink noise: high frequencies have lower power density.

### 5.4.2 Wind Gradients

#### 5.4.2.1 The Nabla Operator

The Nabla operator (partial derivative operator) is given by:

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix}^T$$

Applying the Nabla operator to a scalar field p yields a vector (the gradient): Gradient:

$$\nabla p = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} p = \begin{bmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \\ \frac{\partial p}{\partial z} \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \text{grad}(p)$$

The application of the Nabla operator to a vector field V yields – depending on the type of the product – a scalar (divergence), a vector (rotation) or a matrix (Jacobian matrix): Divergence (scalar product, inner product):

$$\nabla \cdot \boldsymbol{V} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \boldsymbol{V} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = u_x + v_y + w_z = \operatorname{div}\left(\boldsymbol{V}\right)$$

Rotation (cross product):

$$\nabla \times \boldsymbol{V} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \boldsymbol{V} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{bmatrix} = \begin{bmatrix} w_y - v_z \\ u_z - w_x \\ v_x - u_y \end{bmatrix} = \operatorname{rot}(\boldsymbol{V})$$

Jacobian (dyadic product, outer product):

$$\nabla \cdot \mathbf{V}^{T} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \mathbf{V}^{T} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}^{T} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} u & v & w \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} u_{x} & v_{x} & w_{x} \\ u_{y} & v_{y} & w_{y} \\ u_{z} & v_{z} & w_{z} \end{bmatrix}$$

Source-freeness:

$$\operatorname{div}\left(\boldsymbol{V}\right) = u_x + v_y + w_z = 0$$

Rotation-freeness (irrotationality):

$$\operatorname{rot}\left(\boldsymbol{V}\right) = \begin{bmatrix} w_y - v_z \\ u_z - w_x \\ v_x - u_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

#### Wind Gradients 5.4.2.2



increasing in x direction)

(b)  $v_{Wy}$  (wind in y direction increasing in y direction)

(c)  $w_{Wz}$  (wind in z direction increasing in z direction)

Figure 5.7: Wind gradients in each Cartesian direction.



Figure 5.8: Horizontal, source-free wind field  $(u_{Wx} = -v_{Wy})$ .



Figure 5.9: Horizontal, rotation-free wind field  $(u_{Wy} = v_{Wx})$ .



Figure 5.10: Horizontal, rotating wind field  $(u_{Wy} = -v_{Wx})$ .

Description of the wind component caused by wind shear using the Jacobian matrix (shear tensor):

$$\begin{aligned} \boldsymbol{V_{W}}_{Shear} &= \begin{bmatrix} u_W \\ v_W \\ w_W \end{bmatrix}_{Shear} = \left( \nabla \cdot \boldsymbol{V_W^T} \right)^T \cdot \boldsymbol{s} = \begin{bmatrix} u_{Wx} & v_{Wx} & w_{Wx} \\ u_{Wy} & v_{Wy} & w_{Wy} \\ u_{Wz} & v_{Wz} & w_{Wz} \end{bmatrix}^T \boldsymbol{s} \\ &= \begin{bmatrix} u_{Wx} & u_{Wy} & u_{Wz} \\ v_{Wx} & v_{Wy} & v_{Wz} \\ w_{Wx} & w_{Wy} & w_{Wz} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u_{Wx} \cdot x + u_{Wy} \cdot y + u_{Wz} \cdot z \\ v_{Wx} \cdot x + v_{Wy} \cdot y + v_{Wz} \cdot z \\ w_{Wx} \cdot x + w_{Wy} \cdot y + w_{Wz} \cdot z \end{bmatrix} \end{aligned}$$

#### 5.4.3 Flight in a Stationary Wind Field

The flight path velocity of an aircraft is a function of the surrounding wind velocity. At every point:

$$V_K = V_A + V_W$$
 (vector sum)

Consider a simply manoeuvre in a steady wind field, where the aircraft flies a steady, horizontal turn without side-slip angle relative to the air. All aerodynamic variables (angle of attack, lift,  $\ldots$ ) are constant.



Figure 5.11: Flight in a stationary wind field (trochoid, cycloid).

The flight-path velocity vector is always oriented tangentially to the trajectory.

- Point A The "reversal point" of the "circle" with respect to the air, at  $x_g = 0$ . The aerodynamic velocity vector points exactly in the  $x_g$  direction.
- Point B the aircraft flies exactly in negative  $y_g$  direction, against the wind. The magnitude (norm) of the flight-path velocity vector (flight-path speed) is minimum.

- Point C the aircraft flies exactly in positive  $y_g$  direction, with the wind. The magnitude of the flight-path velocity vector is maximum.
- Point D reversal point of the trochoid with respect to the Earth. The flight-path velocity vector points exactly in the  $x_g$  direction.

#### 5.4.3.1 Flight-path Speed and Energy

Between points B and C, the magnitude of the flight-path velocity vector increases; the aircraft thus accelerates relative to the Earth, which is assumed to be at rest. Between C and E the aircraft decelerates.

With respect to the air, the aircraft flies a circle. The aerodynamic velocity vector therefore rotates with the constant yaw angle derivative:

$$\boldsymbol{V}_{Ag} = \begin{bmatrix} V_A \sin\left(\dot{\Psi}t\right) \\ V_A \cos\left(\dot{\Psi}t\right) \end{bmatrix}$$

The wind comes from the west and therefore has only a  $y_g$  component:

$$\boldsymbol{V_{Wg}} = \begin{bmatrix} 0\\ V_W \end{bmatrix}$$

The flight-path velocity vector in the Earth-fixed axis system results from the vector sum:

$$\boldsymbol{V}_{\boldsymbol{K}\boldsymbol{g}} = \boldsymbol{V}_{\boldsymbol{A}\boldsymbol{g}} + \boldsymbol{V}_{\boldsymbol{W}\boldsymbol{g}} = \begin{bmatrix} V_A \sin\left(\dot{\boldsymbol{\Psi}}t\right) \\ V_A \cos\left(\dot{\boldsymbol{\Psi}}t\right) \end{bmatrix} + \begin{bmatrix} 0 \\ V_W \end{bmatrix} = \begin{bmatrix} V_A \sin\left(\dot{\boldsymbol{\Psi}}t\right) \\ V_A \cos\left(\dot{\boldsymbol{\Psi}}t\right) + V_W \end{bmatrix}$$

The magnitude of the flight-path velocity is a periodic function of time:

$$V_{Kg} = |\mathbf{V}_{Kg}| = \sqrt{\left(V_A \sin\left(\dot{\Psi}t\right)\right)^2 + \left(V_W + V_A \cos\left(\dot{\Psi}t\right)\right)^2} \\ = \sqrt{V_A^2 \sin^2\left(\dot{\Psi}t\right) + V_W^2 + 2V_W V_A \cos\left(\dot{\Psi}t\right) + V_A^2 \cos^2\left(\dot{\Psi}t\right)} \\ = \sqrt{V_A^2 + V_W^2 + 2V_W V_A \cos\left(\dot{\Psi}t\right)}$$

If the flight-path speed of the aircraft changes in the course of the trochoid, the kinetic energy cannot be constant either:

$$E_{kin} = \frac{1}{2}mV_{Kg}^2 \neq const.$$

However, since the potential energy of the altitude remains constant during horizontal curved flight, the energy must be taken directly from the surrounding wind field or given off to it.

# 5.5 Kinetics

The aircraft motion has six degrees of freedom, the three **translational** degrees of freedom:

- forward/backward
- right/left
- up/down

and the three **rotational** degrees of freedom:

- roll (about the x axis)
- pitch (about the y axis)
- yaw (about the z axis)

Each degree of freedom is described by two states (speed and position, or angular speed and attitude angle)  $\Rightarrow$  12 states in total.

These 12 states can be combined into four three-dimensional state vectors:

Angular velocity vector:

$$\boldsymbol{\Omega}_{\boldsymbol{K}} = \begin{bmatrix} p_K \\ q_K \\ r_K \end{bmatrix}$$

Attitude vector:

$$oldsymbol{\Phi} = egin{bmatrix} \Phi \ \Theta \ \Psi \end{bmatrix}$$

Flight-path velocity vector:

$$\boldsymbol{V_K} = \begin{bmatrix} u_K \\ v_K \\ w_K \end{bmatrix}$$

Position vector:

$$oldsymbol{s} = egin{bmatrix} x \ y \ z \end{bmatrix}$$

The entire kinetics is then described by a system of four coupled, non-linear vector differential equations:

$$egin{aligned} \dot{mathcal{D}}_{K} &= m{f}_{\Omega}\left(m{Q},\ m{\Omega}_{K}
ight) \ \dot{m{\Phi}} &= m{f}_{m{\Phi}}\left(m{\Omega}_{K},\ m{\Phi}
ight) \ \dot{m{V}}_{K} &= m{f}_{V}\left(m{R},\ m{V}_{K},\ m{\Omega}_{K},\ m{\Phi}
ight) \ \dot{m{s}} &= m{f}_{s}\left(m{V}_{K},\ m{\Phi}
ight) \end{aligned}$$



Figure 5.12: General kinetics of aircraft motion (6 degrees of freedom).

### 5.5.1 Differential Equation for the Position Vector

"Velocity is the change in position over time":

$$\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t} = \boldsymbol{V}_{\boldsymbol{K}}$$

Expressed in the Earth-fixed axis system:

$$\frac{\mathrm{d}\boldsymbol{s_g}}{\mathrm{d}t} = \boldsymbol{V_{Kg}}$$

Transformation of the flight-path velocity:

$$\frac{\mathrm{d}\boldsymbol{s_g}}{\mathrm{d}t} = \boldsymbol{M_{gf}}\boldsymbol{V_{Kf}}$$

Use the derivative point for the direct derivative in the geodetic (inertial) axis system:

$$\dot{s}_g = M_{gf} V_{Kf}$$

### 5.5.2 Differential Equation for the Attitude Vector

"The angular velocity is the change in the attitude over time":

$$\frac{\mathrm{d}\boldsymbol{\varPhi}}{\mathrm{d}t} = \boldsymbol{\varOmega}_{\boldsymbol{K}}$$

Expressed in the body-fixed axis system:

$$\left(\frac{\mathrm{d}\boldsymbol{\varPhi}}{\mathrm{d}t}\right)_f = \boldsymbol{\varOmega}_{\boldsymbol{K}\boldsymbol{f}}$$

*If* all angle components of the attitude vector were now to rotate about the body-fixed axes, *then*:

$$\left(\frac{\mathrm{d}\boldsymbol{\varPhi}}{\mathrm{d}t}\right)_f$$

 $\begin{bmatrix} \dot{\Phi} \\ \dot{\Theta} \\ \dot{\Psi} \end{bmatrix}$ 

This term would contain the time derivatives of the Euler angles:

The differential equation of the angle of rotation is thus complete.

Unfortunately, however,  $\Psi$  does not rotate about the  $z_f$  axis, but about the  $z_g$  axis and  $\Theta$  does not rotate about the  $y_f$  axis, but about the  $k_2$  node axis. The corresponding two angular derivatives must therefore first be transformed individually into the body-fixed axis system:

$$\begin{split} \left(\frac{\mathrm{d}\boldsymbol{\Phi}}{\mathrm{d}t}\right)_{f} &= \begin{bmatrix} \dot{\boldsymbol{\Phi}}\\ \boldsymbol{0}\\ \boldsymbol{0} \end{bmatrix} \\ &+ \begin{bmatrix} 1 & 0 & 0\\ 0 & \cos\boldsymbol{\Phi} & \sin\boldsymbol{\Phi}\\ \boldsymbol{0} & -\sin\boldsymbol{\Phi} & \cos\boldsymbol{\Phi} \end{bmatrix} \begin{bmatrix} 0\\ \dot{\boldsymbol{\Theta}}\\ \boldsymbol{0} \end{bmatrix} \\ &+ \begin{bmatrix} 1 & 0 & 0\\ 0 & \cos\boldsymbol{\Phi} & \sin\boldsymbol{\Phi}\\ \boldsymbol{0} & -\sin\boldsymbol{\Phi} & \cos\boldsymbol{\Phi} \end{bmatrix} \begin{bmatrix} \cos\boldsymbol{\Theta} & 0 & -\sin\boldsymbol{\Theta}\\ \boldsymbol{0} & 1 & 0\\ \sin\boldsymbol{\Theta} & \boldsymbol{0} & \cos\boldsymbol{\Theta} \end{bmatrix} \begin{bmatrix} 0\\ 0\\ \dot{\boldsymbol{\psi}} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -\sin\boldsymbol{\Theta}\\ 0 & \cos\boldsymbol{\Phi} & \sin\boldsymbol{\Phi}\cos\boldsymbol{\Theta}\\ \boldsymbol{0} & -\sin\boldsymbol{\Phi} & \cos\boldsymbol{\Phi}\cos\boldsymbol{\Theta} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\Phi}}\\ \dot{\boldsymbol{\Theta}}\\ \dot{\boldsymbol{\psi}} \end{bmatrix} \end{split}$$

The differential equation of the attitude vector is then:

$$\begin{bmatrix} 1 & 0 & -\sin\Theta \\ 0 & \cos\Phi & \sin\Phi\cos\Theta \\ 0 & -\sin\Phi & \cos\Phi\cos\Theta \end{bmatrix} \begin{bmatrix} \dot{\Phi} \\ \dot{\Theta} \\ \dot{\Psi} \end{bmatrix} = \boldsymbol{\Omega}_{Kf} = \begin{bmatrix} p_{Kf} \\ q_{Kf} \\ r_{Kf} \end{bmatrix}$$

To solve for the derivative vector, the transformation matrix must be inverted. Unfortunately, it is not orthogonal (because the Euler angles rotate about axes that are not perpendicular to each other) and therefore cannot be inverted simply by transposing:

$$\dot{\boldsymbol{\Phi}} = \begin{bmatrix} \dot{\boldsymbol{\Phi}} \\ \dot{\boldsymbol{\Theta}} \\ \dot{\boldsymbol{\Psi}} \end{bmatrix} = \begin{bmatrix} 1 & \sin \boldsymbol{\Phi} \tan \boldsymbol{\Theta} & \cos \boldsymbol{\Phi} \tan \boldsymbol{\Theta} \\ 0 & \cos \boldsymbol{\Phi} & -\sin \boldsymbol{\Phi} \\ 0 & \sin \boldsymbol{\Phi}/\cos \boldsymbol{\Theta} & \cos \boldsymbol{\Phi}/\cos \boldsymbol{\Theta} \end{bmatrix} \boldsymbol{\Omega}_{Kf} = \boldsymbol{M}_{\boldsymbol{\Phi}f} \cdot \boldsymbol{\Omega}_{Kf}$$
(5.1)

## 5.5.3 Differential Equation of the Flight-Path Velocity Vector

Momentum theorem: "The force  $\boldsymbol{R}$  is the temporal change of the momentum  $\boldsymbol{P}$ ":

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = \boldsymbol{R}$$

Expressed in the body-fixed axis system:

$$\left(\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t}\right)_f = \boldsymbol{R}_f$$

Problem: The body-fixed axis system in which the change in momentum

.

$$\left(\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t}\right)_f$$

is described, is not an inertial system, but rotates with the flight-path angular velocity  $\Omega_K$  with respect to the Earth, which is assumed to be at rest. As shown in section 5.5.3.1, the rotation of the body-fixed axis system must therefore be taken into account in the so-called Euler term (cross product) for the inertial derivation of the momentum:

$$\dot{P}_f + arOmega_{Kf} imes P_f = R_f$$

"Momentum is mass times velocity":

$$(mV_{Kf})$$
 +  $\Omega_{Kf} \times (mV_{Kf}) = R_f$ 

Product rule of differentiation:

$$\dot{m}V_{Kf} + m\dot{V}_{Kf} + \Omega_{Kf} \times (mV_{Kf}) = R_f$$

Neglecting the change in mass:

$$mV_{Kf} + \Omega_{Kf} \times (mV_{Kf}) = R_f$$

Factoring out the constant, scalar mass:

$$m\left(\dot{V}_{Kf}+\boldsymbol{\Omega}_{Kf}\times V_{Kf}\right)=\boldsymbol{R}_{f}$$

The total force from the engine, aerodynamics and weight is given by:

$$m\left(\dot{V}_{Kf} + \boldsymbol{\Omega}_{Kf} \times \boldsymbol{V}_{Kf}\right) = \boldsymbol{R}_{f}^{F} + \boldsymbol{R}_{f}^{A} + \boldsymbol{G}_{f}$$

Transformation of weight and aerodynamic force:

$$m\left(\dot{V}_{Kf} + \boldsymbol{\Omega}_{Kf} \times \boldsymbol{V}_{Kf}\right) = \boldsymbol{R}_{f}^{F} + \boldsymbol{M}_{fa}\boldsymbol{R}_{a}^{A} + \boldsymbol{M}_{fg}\boldsymbol{G}_{g}$$

Solve for the derivative:

$$\dot{V}_{Kf} = rac{1}{m} \left( oldsymbol{R}_{f}^{F} + oldsymbol{M}_{fa} oldsymbol{R}_{a}^{A} + oldsymbol{M}_{fg} oldsymbol{G}_{g} 
ight) - oldsymbol{\Omega}_{Kf} imes oldsymbol{V}_{Kf}$$

"Canceling" out the mass from the weight:

$$\dot{V}_{Kf} = rac{1}{m} \left( oldsymbol{R}_{oldsymbol{f}}^{F} + oldsymbol{M}_{oldsymbol{fa}} oldsymbol{R}_{oldsymbol{a}}^{A} 
ight) + oldsymbol{M}_{oldsymbol{fg}} oldsymbol{g}_{g} - oldsymbol{\Omega}_{Kf} imes oldsymbol{V}_{Kf}$$

#### 5.5.3.1 Derivative of a Vector in a Rotating Axis System

In the following, the geodetic axis system is understood as an inertial (stationary, spacefixed) axis system and the body-fixed axis system is used as an example of a non-inertial (rotating) axis system.

The inertial derivative (superscript g) of a vector V expressed in the body-fixed axis system (subscript f) is obtained by transforming the vector from the body-fixed axis system into the geodetic axis system, deriving it inertially there and then transforming it back into the aircraft axis system:

$$\left(\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}\right)_{f}^{g} = \boldsymbol{M}_{fg}\frac{\mathrm{d}(\boldsymbol{M}_{gf}\boldsymbol{V}_{f})}{\mathrm{d}t}$$

Using the Product Rule:

$$\left(\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}\right)_{f}^{g} = \boldsymbol{M}_{fg} \left(\boldsymbol{M}_{gf} \left(\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}\right)_{f}^{f} + \frac{\mathrm{d}\left(\boldsymbol{M}_{gf}\right)}{\mathrm{d}t}\boldsymbol{V}_{f}\right)$$

Multiply out:

$$\left(\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}\right)_{f}^{g} = \boldsymbol{M}_{fg}\boldsymbol{M}_{gf}\left(\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}\right)_{f}^{f} + \boldsymbol{M}_{fg}\frac{\mathrm{d}\left(\boldsymbol{M}_{gf}\right)}{\mathrm{d}t}\boldsymbol{V}_{f}$$

Combine matrices and use derivative points:

$$\left(\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}\right)_{f}^{g}=\dot{\boldsymbol{V}}_{f}+\boldsymbol{M}_{fg}\dot{\boldsymbol{M}}_{gf}\boldsymbol{V}_{f}$$

Therefore the term

$$\left(\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}\right)_{f}^{f} = \dot{\boldsymbol{V}}_{f}$$

is the derivative of the vector performed component-wise directly in the body-fixed axis system, again expressed in the body-fixed axis system. The time derivative of the transformation matrix:

$$\frac{\mathrm{d}\left(\boldsymbol{M}_{gf}\right)}{\mathrm{d}t} = \dot{\boldsymbol{M}}_{gf}$$

is also performed individually for each element of the matrix.

In a somewhat longer derivation (Matlab file: http://buchholz.hs-bremen.de/rtfr/skript/euler\_term.mlx) the matrix product containing the Euler angle derivatives

$$M_{fg}M_{gf}$$

can be combined to a cross product with the vector of the body-fixed flight-path angular velocity:

$$\left(\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}\right)_{f}^{g} = \dot{\boldsymbol{V}}_{f} + \boldsymbol{\Omega}_{Kf} \times \boldsymbol{V}_{f}$$

The cross product  $\Omega_{Kf} \times V_f$  is sometimes called the "Euler term".

5.5.3.2 Example: horizontal (tethered) turn (without banking, without slipping )



Figure 5.13: Rotation of the body-fixed axis system and the flight-path velocity vector during horizontal turn.

Since the flight-path velocity vector  $V_K$  (tangential to the flight-path) rotates together with the aircraft (and thus also with the body-fixed axis system) in a horizontal, bank-free and slip-free turn, it always points in the  $x_f$  direction.

It is therefore constant when expressed in the body-fixed axis system:

$$\boldsymbol{V_{Kf}} = \begin{bmatrix} \boldsymbol{u_{Kf}} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}$$

Therefore, the derivative carried out component-wise directly in the body-fixed axis system vanishes:

$$\left(\frac{\mathrm{d}\boldsymbol{V}_{\boldsymbol{K}}}{\mathrm{d}t}\right)_{f}^{f} = \dot{\boldsymbol{V}}_{\boldsymbol{K}f} = \begin{bmatrix} 0\\0\\0\end{bmatrix}$$

Since the flight-path lies in the Earth's horizontal plane, the angular velocity vector can only have a z component which is equal to the temporal change of the yaw angle in the negative direction of rotation:

$$\boldsymbol{\Omega}_{Kf} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ -\dot{\boldsymbol{\Psi}} \end{bmatrix}$$

The complete inertial flight-path velocity vector derivative is then:

$$\left(\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}\right)_{f}^{g} = \dot{\boldsymbol{V}}_{f} + \boldsymbol{\Omega}_{Kf} \times \boldsymbol{V}_{f} = \begin{bmatrix} 0\\0\\0\\-\dot{\Psi} \end{bmatrix} + \begin{bmatrix} 0\\0\\-\dot{\Psi} \end{bmatrix} \times \begin{bmatrix} u_{Kf}\\0\\0\\0 \end{bmatrix} = \begin{bmatrix} 0\\-\dot{\Psi} \cdot u_{Kf}\\0\\0 \end{bmatrix}$$

The term  $-\dot{\Psi} \cdot u_{Kf}$  corresponds, physically correct, exactly to the centripetal acceleration which acts in the negative  $y_f$  direction and keeps the aircraft on its circular path.

# 5.5.4 Differential equation of the flight-path angular velocity vector

The derivation of the angular speed differential equation is analogous to the derivation of the velocity differential equation. Only the forces have to be replaced by the moments, the momentum by the angular momentum, the velocity by the angular velocity and the scalar mass by the tensor of the moments of inertia:

Angular momentum theorem: "The moment Q is the temporal change of the angular momentum D":

$$\frac{\mathrm{d}\boldsymbol{D}}{\mathrm{d}t} = \boldsymbol{Q}$$

Expressed in the body-fixed axis system:

$$\left(\frac{\mathrm{d}\boldsymbol{D}}{\mathrm{d}t}\right)_f = \boldsymbol{Q}_f$$

Inertial derivation:

$$\dot{D}_f + arOmega_{Kf} imes D_f = Q_f$$

"Angular momentum is inertia tensor times angular velocity":

$$(\boldsymbol{I_f}\cdot\boldsymbol{\varOmega_{Kf}})^{\cdot}+\boldsymbol{\varOmega_{Kf}} imes(\boldsymbol{I_f}\cdot\boldsymbol{\varOmega_{Kf}})=\boldsymbol{Q_f}$$

Constant inertia tensor:

$$oldsymbol{I}_f\cdot\dot{oldsymbol{\Omega}}_{Kf}+oldsymbol{\Omega}_{Kf} imes(oldsymbol{I}_f\cdotoldsymbol{\Omega}_{Kf})=oldsymbol{Q}_f$$

Total moment from engines and aerodynamics

$$oldsymbol{I}_f\cdot\dot{oldsymbol{\Omega}}_{Kf}+oldsymbol{\Omega}_{Kf} imes(oldsymbol{I}_f\cdotoldsymbol{\Omega}_{Kf})=oldsymbol{Q}_f^F+oldsymbol{Q}_f^A$$

Transformation of the aerodynamic moment vector:

$$oldsymbol{I}_f\cdot\dot{oldsymbol{\Omega}}_{Kf}+oldsymbol{\Omega}_{Kf} imes(oldsymbol{I}_f\cdotoldsymbol{\Omega}_{Kf})=oldsymbol{Q}_f^F+oldsymbol{M}_{fa}oldsymbol{Q}_a^A$$

Solve for the derivative:

$$\dot{\boldsymbol{\Omega}}_{Kf} = \boldsymbol{I}_{f}^{-1} \cdot \left( \boldsymbol{Q}_{f}^{F} + \boldsymbol{M}_{fa} \boldsymbol{Q}_{a}^{A} - \boldsymbol{\Omega}_{Kf} imes \left( \boldsymbol{I}_{f} \cdot \boldsymbol{\Omega}_{Kf} 
ight) 
ight)$$

Unfortunately, the inertia tensor – as a matrix – cannot be excluded from the cross product. For the same reason, its reciprocal value is calculated by regular matrix inversion.

#### 5.5.4.1 The Inertia Tensor

The inertia tensor describes, analogously to the mass in translational motions, the inertia with which the system resists a change in angular motion. However, while the mass as a scalar quantity is the same in all translational directions, the angular inertias differ depending on the axis of rotation considered.

In the body-fixed axis system, the (symmetric) inertia tensor is given by:

$$\boldsymbol{I_f} = \begin{bmatrix} I_{xf} & -I_{xyf} & -I_{xzf} \\ -I_{xyf} & I_{yf} & -I_{yzf} \\ -I_{xzf} & -I_{yzf} & I_{zf} \end{bmatrix}$$

On its main diagonal are the moments of inertia: are:

$$I_{xf} = \int \left(y_f^2 + z_f^2\right) \cdot dm$$
$$I_{yf} = \int \left(x_f^2 + z_f^2\right) \cdot dm$$
$$I_{zf} = \int \left(x_f^2 + y_f^2\right) \cdot dm$$

In other words, a moment of inertia is the sum (integral) of all infinitesimally small mass particles multiplied by the square of their respective lever arm (Pythagorean distance from the corresponding axis of rotation).

The non-diagonal elements of the inertia tensor are called products of inertia:

$$I_{xyf} = \int (x_f \cdot y_f) \cdot dm$$
$$I_{xzf} = \int (x_f \cdot z_f) \cdot dm$$
$$I_{yzf} = \int (y_f \cdot z_f) \cdot dm$$

In the case of a product of inertia, the mass particles are formally multiplied by two lever arms (position coordinates) each, which has the consequence that for symmetrical aircraft  $I_{xyf}$  and  $I_{yzf}$  disappear. If the  $x_{f}$ - $z_{f}$  plane represents a plane of symmetry for an aircraft, for every mass particle on the right side of the plane of symmetry there is a corresponding (identical) mass particle on the left side. Both differ only by the sign of their  $y_{f}$  coordinate (right positive, left negative), so that the integral (the sum) always disappears if the integrand contains a  $y_{f}$  factor: Symmetrical aircraft:

$$I_{xyf} = I_{yzf} = 0$$

In addition to the moments of inertia, the inertia tensor then only contains the product of inertia  $I_{xzf}$ :

$$\boldsymbol{I_f} = \begin{bmatrix} I_{xf} & 0 & -I_{xzf} \\ 0 & I_{yf} & 0 \\ -I_{xzf} & 0 & I_{zf} \end{bmatrix}$$
(5.2)

If the body-fixed axes are defined in the direction of the main inertia axes, then (and only then) the last deviation moment also disappears:

$$I_{xzf} = 0$$
Effect of the Deviation Moments Newton's Second Law:

$$\boldsymbol{F} = m \cdot \boldsymbol{a}$$

Can be transformed for rotary motion; the force F becomes the moment Q, the acceleration a becomes the angular acceleration  $\dot{\Omega}$  and the scalar mass m becomes the inertia tensor I:

$$oldsymbol{Q} = oldsymbol{I} \cdot \dot{oldsymbol{\Omega}}$$

Solving for the angular acceleration gives:

$$\dot{\boldsymbol{\Omega}} = \boldsymbol{I}^{-1} \boldsymbol{Q}. \tag{5.3}$$

The inversion of the inertia tensor can still be performed relatively clearly in an analytical manner. The inverse of a matrix can be calculated from the quotient of its adjugate matrix and its determinant:

$$I^{-1} = rac{1}{|I|} \cdot I_{adj}$$

Using eqn. (5.2) the result is (with the index f omitted for the sake of clarity):

$$\begin{bmatrix} I_x & 0 & -I_{xz} \\ 0 & I_y & 0 \\ -I_{xz} & 0 & I_z \end{bmatrix}^{-1} = \frac{1}{I_x I_y I_z - I_{xz}^2 I_y} \begin{bmatrix} I_y I_z & 0 & I_{xz} I_y \\ 0 & I_x I_z - I_{xz}^2 & 0 \\ I_{xz} I_y & 0 & I_x I_y \end{bmatrix}$$
$$= \begin{bmatrix} \frac{I_z}{I_x I_z - I_{xz}^2} & 0 & \frac{I_{xz}}{I_x I_z - I_{xz}^2} \\ 0 & \frac{1}{I_y} & 0 \\ \frac{I_{xz}}{I_x I_z - I_{xz}^2} & 0 & \frac{I_x}{I_x I_z - I_{xz}^2} \end{bmatrix}$$

Now, if this inverse inertia tensor is used to establish the relationship between angular acceleration  $\dot{\Omega}$  and angular moment Q in eqn. (5.3).

$$\begin{aligned} \dot{\boldsymbol{\Omega}} &= \boldsymbol{I}^{-1} \cdot \boldsymbol{Q} \\ \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} &= \begin{bmatrix} \frac{I_z}{I_x I_z - I_{xz}^2} & 0 & \frac{I_{xz}}{I_x I_z - I_{xz}^2} \\ 0 & \frac{1}{I_y} & 0 \\ \frac{I_{xz}}{I_x I_z - I_{xz}^2} & 0 & \frac{I_x}{I_x I_z - I_{xz}^2} \end{bmatrix} \begin{bmatrix} L \\ M \\ N \end{bmatrix} \end{aligned}$$

The first line shows that a roll acceleration  $\dot{p}$  is not only generated by a roll moment L, but that a yaw moment N also contributes to the total roll acceleration via the deviation moment  $I_{xz}$ :

$$\dot{p} = \frac{I_z}{I_x I_z - I_{xz}^2} L + \frac{I_{xz}}{I_x I_z - I_{xz}^2} N$$

Argued the other way round, a pure yaw moment not only causes a yaw acceleration but also a parasitic roll acceleration.



Figure 5.14: With a pure yaw moment, the aircraft also reacts with a roll acceleration (in addition to the yaw acceleration) because of the "inertia" of the vertical stabilizer.

## 5.6 Quaternions

The non-orthogonal transformation matrix in the differential equation of the attitude vector (5.1) contains the following quotients:

$$\sin\Phi\tan\Theta = \frac{\sin\Phi\sin\Theta}{\cos\Theta}, \quad \cos\Phi\tan\Theta = \frac{\cos\Phi\sin\Theta}{\cos\Theta}, \quad \frac{\sin\Phi}{\cos\Theta}, \quad \frac{\cos\Phi}{\cos\Theta}, \quad \frac{\Theta}{\cos\Theta}, \quad \frac{\Theta}{\cos\Phi}, \quad \frac$$

The denominator disappears for a pitch angle  $\Theta$  of  $\pm \frac{\pi}{2}$ . If additionally the roll angle  $\Phi$  is a multiple of  $\frac{\pi}{2}$ , the corresponding quotient is indefinite  $\begin{pmatrix} 0\\0 \end{pmatrix}$ , otherwise the quotients take on infinitely large values. A simulation terminates in all cases with an error message.

From a physical perspective, the indeterminacy problem manifests itself, for example, in an airplane oriented vertically downwards  $\left(\Theta = -\frac{\pi}{2}\right)$ , through the coincidence of the geodetic z axis and the body-fixed x axis.

Both a "yaw" with  $\dot{\Psi}$  about the  $z_g$  axis as well as a "roll" with  $\dot{\Phi}$  about the  $x_f$  axis now lead to the same motion about the vertical aircraft longitudinal axis (gimbal lock).

To solve the problem, one can use the four components of a quaternion  $[a \ b \ c \ d]$  as state variables in the differential equation of the attitude vector instead of the three Euler angles  $[\Phi \ \Theta \ \Psi]$ .

#### 5.6.1 Properties of Quaternions

Quaternions are – similar to the complex numbers – an extension of the real numbers. While a complex number z consists of a real part a and a **scalar** imaginary part b ( $z = a + b \cdot i$ ), a quaternion Z has a real part a and a **vectorial** imaginary part [b c d], whose components are each multiplied by their own imaginary unit i, j and k:

$$Z = a + b \cdot \mathbf{i} + c \cdot \mathbf{j} + d \cdot \mathbf{k}$$

The imaginary units are defined in the same way as complex numbers:

$$i^2 = j^2 = k^2 = -1$$

In addition, the product of two different imaginary units results in the third, and it is anti-commutative (change of sign when the order is reversed):

$$\begin{aligned} i \cdot j &= k & j \cdot k = i & k \cdot i = j \\ j \cdot i &= -k & k \cdot j = -i & i \cdot k = -j \end{aligned}$$

The sum of two quaternions is calculated component-wise:

$$Z_1 + Z_2 = (a_1 + b_1 \mathbf{i} + c_1 \mathbf{j} + d_1 \mathbf{k}) + (a_2 + b_2 \mathbf{i} + c_2 \mathbf{j} + d_2 \mathbf{k})$$
  
=  $(a_1 + a_2) + (b_1 + b_2) \mathbf{i} + (c_1 + c_2) \mathbf{j} + (d_1 + d_2) \mathbf{k}$ 

whereas with the quaternion product the signs of the products of the imaginary units must be taken into account:

$$Z_1 \cdot Z_2 = (a_1 + b_1 \mathbf{i} + c_1 \mathbf{j} + d_1 \mathbf{k}) \cdot (a_2 + b_2 \mathbf{i} + c_2 \mathbf{j} + d_2 \mathbf{k})$$
  
=  $(a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2)$   
+  $(a_1 b_2 + b_1 a_2 + c_1 d_2 - d_1 c_2) \mathbf{i}$   
+  $(a_1 c_2 - b_1 d_2 + c_1 a_2 + d_1 b_2) \mathbf{j}$   
+  $(a_1 d_2 + b_1 c_2 - c_1 b_2 + d_1 a_2) \mathbf{k}$ 

The conjugate quaternion  $\overline{Z}$  results – as with the complex numbers – from a negative sign in the imaginary part:

$$\overline{Z} = \overline{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}} = a - (b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$$
(5.4)

The product of a quaternion with its conjugate is purely real:

$$Z \cdot \overline{Z} = (a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) \cdot (a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k})$$
$$= (aa + bb + cc + dd)$$
$$+ (-ab + ba - cd + dc) \mathbf{i}$$
$$+ (-ac + bd + ca - db) \mathbf{j}$$
$$+ (-ad - bc + cb + da) \mathbf{k}$$
$$= a^2 + b^2 + c^2 + d^2$$

and corresponds to the norm squared (magnitude squared)  $|Z|^2$  of the quaternion:

$$|Z| = \sqrt{Z \cdot \overline{Z}} = \sqrt{a^2 + b^2 + c^2 + d^2}$$

Any quaternion with a non-zero length can be transformed into its unit quaternion by dividing it by its magnitude:

$$Z^{0} = \frac{Z}{|Z|} = \frac{a}{|Z|} + \frac{b}{|Z|}\mathbf{i} + \frac{c}{|Z|}\mathbf{j} + \frac{d}{|Z|}\mathbf{k}$$
(5.5)

## 5.6.2 Calculation of the Quaternion from the Angle of Rotation and the Axis of Rotation

In eqn. (4.1) the transformation matrix  $M_{fg}$  is defined, which consists of trigonometric functions of the Euler angles  $\Phi$ ,  $\Theta$ , and  $\Psi$ , and which transforms a vector  $\boldsymbol{v} = [x \ y \ z]^T$  from the geodetic (index g) to the body-fixed (index f) axis system:

 $\begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} = \begin{bmatrix} \cos\Theta\cos\Psi & \cos\Theta\sin\Psi & -\sin\Theta \\ \sin\Phi\sin\Theta\cos\Psi - \cos\Phi\sin\Psi & \sin\Phi\sin\Theta\sin\Psi + \cos\Phi\cos\Psi & \sin\Phi\cos\Theta \\ \cos\Phi\sin\Theta\cos\Psi + \sin\Phi\sin\Psi & \cos\Phi\sin\Theta\sin\Psi - \sin\Phi\cos\Psi & \cos\Phi\cos\Theta \end{bmatrix} \begin{bmatrix} x_g \\ y_g \\ z_g \end{bmatrix}$ (5.6)

The same transformation can also be realised with a quaternion:



Figure 5.15: Total rotation from the geodetic to the body-fixed axis system with the angle  $\Xi$  about the rotation axis n

For this purpose, the three individual rotations with  $\Psi$ ,  $\Theta$ , and  $\Phi$  shown in figure 4.8, which are necessary to transfer the geodetic axis system into the body-fixed axis system, are combined into the total rotation with the angle  $\Xi$  about the rotation axis  $\boldsymbol{n} = [n_x n_y n_z]^T$ shown in figure 5.15. (The rotation axis has the same coordinates in both axis systems, because the body-fixed axis system rotates exactly about the rotation axis and the coordinates of the rotation axis do not change).

Provided that the rotation axis vector  $\mathbf{n}$  is a unit vector (i. e. its length has been normalised to one according to eqn. (5.5)), the corresponding unit quaternion  $Z_D$  can be calculated from the angle of rotation and axis of rotation:

$$Z_D = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$
  
=  $\cos\left(\frac{\Xi}{2}\right) + n_x \cdot \sin\left(\frac{\Xi}{2}\right)\mathbf{i} + n_y \cdot \sin\left(\frac{\Xi}{2}\right)\mathbf{j} + n_z \cdot \sin\left(\frac{\Xi}{2}\right)\mathbf{k}.$  (5.7)

## 5.6.3 Calculation of the Angle of Rotation and the Axis of Rotation from the Quaternion

For a given quaternion  $Z_D = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ , the corresponding rotation angle  $\Xi$  and the rotation axis  $\mathbf{n} = [n_x \ n_y \ n_z]^T$  are calculated directly from eqn. (5.7). The real part of the quaternion provides the angle of rotation:

$$a = \cos\left(\frac{\Xi}{2}\right) \quad \Rightarrow \quad \Xi = 2\arccos\left(a\right)$$

With the angle of rotation calculated, the axis of rotation then results from the imaginary part of the quaternion:

$$b\mathbf{i} + c\mathbf{j} + d\mathbf{k} = n_x \cdot \sin\left(\frac{\Xi}{2}\right)\mathbf{i} + n_y \cdot \sin\left(\frac{\Xi}{2}\right)\mathbf{j} + n_z \cdot \sin\left(\frac{\Xi}{2}\right)\mathbf{k}$$
$$\Rightarrow \quad \boldsymbol{n} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} \frac{b}{\sin\left(\frac{\Xi}{2}\right)} \\ \frac{c}{\sin\left(\frac{\Xi}{2}\right)} \\ \frac{d}{\sin\left(\frac{\Xi}{2}\right)} \end{bmatrix}$$

## 5.6.4 Calculation of the Euler Angles from the Transformation Matrix

Equations are obtained for the Euler angles from the individual elements of the transformation matrix in eqn. (5.6):

$$M_{13} = -\sin\Theta \tag{5.8}$$

$$M_{11} = \cos\Theta\cos\Psi \tag{5.9}$$

$$M_{12} = \cos\Theta\sin\Psi \tag{5.10}$$

$$M_{23} = \sin \Phi \cos \Theta \tag{5.11}$$

$$M_{33} = \cos \Phi \cos \Theta \tag{5.12}$$

The pitch angle follows directly from eqn. (5.8):

$$-\sin\Theta = M_{13} \quad \Rightarrow \quad \Theta = -\arcsin M_{13}$$
 (5.13)

From the quotient of eqs (5.10) and (5.9) the yaw angle is calculated

$$\frac{\cos\Theta\sin\Psi}{\cos\Theta\cos\Psi} = \tan\Psi = \frac{M_{12}}{M_{11}} \quad \Rightarrow \quad \Psi = \arctan\left(\frac{M_{12}}{M_{11}}\right) \tag{5.14}$$

and the quotient of eqs (5.11) and (5.12) leads to the bank angle:

$$\frac{\sin\Phi\cos\Theta}{\cos\Phi\cos\Theta} = \tan\Phi = \frac{M_{23}}{M_{33}} \quad \Rightarrow \quad \Phi = \arctan\left(\frac{M_{23}}{M_{33}}\right) \tag{5.15}$$

In order to obtain the full range of angles  $(-\pi \dots \pi)$  for  $\Psi$  and  $\Phi$ , the **atan2()** function – available in most programming languages – must be used, which can also handle the singularities that occur with a "normal" arc tangent when the denominators of eqs (5.14) or (5.15) disappear because an angle is  $\frac{\pi}{2}$ . In some publications it is suggested that after calculating the pitch angle according to eqn. (5.13), it should be inserted into eqs (5.9) - (5.12) in order to calculate  $\Psi$  and  $\Phi$  with arcsin or arccos functions. In this way, however,  $\Psi$  and  $\Phi$  would be incorrectly restricted to the ranges  $(-\frac{\pi}{2} \dots \frac{\pi}{2})$  and  $(0 \dots \pi)$  respectively, since arcsin and arccos are only able to deliver values in these ranges.

### 5.6.5 Calculation of the Transformation Matrix from the Quaternion

The conjugate quaternion  $\overline{Z_D}$  to the quaternion defined in eqn. (5.7) is given by eqn. (5.4):

$$\overline{Z_D} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$$

The vector to be transformed  $\boldsymbol{v}_{g} = [x_{g} \ y_{g} \ z_{g}]^{T}$  is given in the form of a quaternion  $Z_{g}$ :

$$Z_g = 0 + x_g \mathbf{i} + y_g \mathbf{j} + z_g \mathbf{k}$$

The transformation to the body-fixed axis system analogous to eqn. (5.6) is then carried out by means of two quaternion products:

$$Z_{f} = \overline{Z_{D}} \cdot Z_{g} \cdot Z_{D}$$

$$= (a - bi - cj - dk) \cdot (0 + x_{g}i + y_{g}j + z_{g}k) \cdot (a + bi + cj + dk)$$

$$= 0$$

$$+ ((a^{2} + b^{2} - c^{2} - d^{2}) x_{g} + 2 (bc + ad) y_{g} + 2 (bd - ac) z_{g}) i$$

$$+ (2 (bc - ad) x_{g} + (a^{2} - b^{2} + c^{2} - d^{2}) y_{g} + 2 (cd + ab) z_{g}) j$$

$$+ (2 (bd + ac) x_{g} + 2 (cd - ab) y_{g} + (a^{2} - b^{2} - c^{2} + d^{2}) z_{g}) k$$
(5.16)

The quaternion is given by:

$$Z_f = 0 + x_f \mathbf{i} + y_f \mathbf{j} + z_f \mathbf{k}$$

and contains the components of the transformed vector  $\boldsymbol{v}_{f} = [x_{f} y_{f} z_{f}]^{T}$ . eqn. (5.16) can be presented in matrix notation:

$$\begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} = \begin{bmatrix} (a^2 + b^2 - c^2 - d^2) & 2(bc + ad) & 2(bd - ac) \\ 2(bc - ad) & (a^2 - b^2 + c^2 - d^2) & 2(cd + ab) \\ 2(bd + ac) & 2(cd - ab) & (a^2 - b^2 - c^2 + d^2) \end{bmatrix} \begin{bmatrix} x_g \\ y_g \\ z_g \end{bmatrix}$$
(5.17)

The transformation matrix can be calculated directly from the quaternion components:

$$\boldsymbol{M_{fg}} = \begin{bmatrix} (a^2 + b^2 - c^2 - d^2) & 2(bc + ad) & 2(bd - ac) \\ 2(bc - ad) & (a^2 - b^2 + c^2 - d^2) & 2(cd + ab) \\ 2(bd + ac) & 2(cd - ab) & (a^2 - b^2 - c^2 + d^2) \end{bmatrix}$$
(5.18)

If the Euler angles are not explicitly required in a simulation, the transformation matrix  $M_{fg}$  built up from the components of the quaternion from eqn. (5.18) can be used directly in the differential equations of the flight-path velocity and the position, so that no trigonometric functions of the Euler angles have to be calculated and the computational effort is reduced.

#### 5.6.6 Calculation of the Euler angles from the Quaternion

The Euler angles can be calculated directly from the quaternion components by inserting the corresponding elements of the transformation matrix from eqn. (5.18) into eqs (5.13) - (5.15):

$$\Theta = -\arcsin M_{13} = -\arcsin \left(2\left(bd - ac\right)\right) = \arcsin \left(2\left(ac - bd\right)\right) \tag{5.19}$$

$$\Psi = \arctan\left(\frac{M_{12}}{M_{11}}\right) = \arctan\left(\frac{2(bc+ad)}{a^2 + b^2 - c^2 - d^2}\right)$$
(5.20)

$$\Phi = \arctan\left(\frac{M_{23}}{M_{33}}\right) = \arctan\left(\frac{2(cd+ab)}{a^2 - b^2 - c^2 + d^2}\right)$$
(5.21)

Of course, in a numerical implementation of eqs (5.19) - (5.21), the **atan2()** function must also be used here.

#### 5.6.7 Calculation of the Quaternion from the Euler Angles

The transformation from the geodetic to the body-fixed axis system shown in eqn. (5.16) with the aid of the total quaternion  $Z_D$  can also be built up from the inside outwards from the three individual quaternions  $Z_{\Psi}$ ,  $Z_{\Theta}$  and  $Z_{\Phi}$ :

$$Z_{f} = \overline{Z_{\Phi}} \cdot \left(\overline{Z_{\Theta}} \cdot \left(\overline{Z_{\Psi}} \cdot Z_{g} \cdot Z_{\Psi}\right) \cdot Z_{\Theta}\right) \cdot Z_{\Phi}$$

$$= \left(\overline{Z_{\Phi}} \cdot \overline{Z_{\Theta}} \cdot \overline{Z_{\Psi}}\right) \cdot Z_{g} \cdot \left(Z_{\Psi} \cdot Z_{\Theta} \cdot Z_{\Phi}\right)$$

$$= \underbrace{\left(\overline{Z_{\Psi}} \cdot Z_{\Theta} \cdot Z_{\Phi}\right)}_{\overline{Z_{D}}} \cdot Z_{g} \cdot \underbrace{\left(\overline{Z_{\Psi}} \cdot Z_{\Theta} \cdot Z_{\Phi}\right)}_{Z_{D}}$$
(5.22)

The associative law that applies to quaternions is taken into account as well as the fact that the conjugate of a quaternion product is equal to the product of the individual conjugates in reverse order.

The single quaternions are constructed according to eqn. (5.7) from the respective rotation angles  $(\Psi, \Theta, \text{ and } \Phi)$  and the corresponding rotation axes  $(\boldsymbol{n}_{\Psi} = [0 \ 0 \ 1]^T, \ldots)$ :

$$\begin{aligned} Z_{\Psi} &= \cos\left(\frac{\Psi}{2}\right) + 0 \cdot \sin\left(\frac{\Psi}{2}\right) \mathbf{i} + 0 \cdot \sin\left(\frac{\Psi}{2}\right) \mathbf{j} + 1 \cdot \sin\left(\frac{\Psi}{2}\right) \mathbf{k} \\ &= \cos\left(\frac{\Psi}{2}\right) + \sin\left(\frac{\Psi}{2}\right) \mathbf{k} \\ Z_{\Theta} &= \cos\left(\frac{\Theta}{2}\right) + 0 \cdot \sin\left(\frac{\Theta}{2}\right) \mathbf{i} + 1 \cdot \sin\left(\frac{\Theta}{2}\right) \mathbf{j} + 0 \cdot \sin\left(\frac{\Theta}{2}\right) \mathbf{k} \\ &= \cos\left(\frac{\Theta}{2}\right) + \sin\left(\frac{\Theta}{2}\right) \mathbf{j} \\ Z_{\Phi} &= \cos\left(\frac{\Phi}{2}\right) + 1 \cdot \sin\left(\frac{\Phi}{2}\right) \mathbf{i} + 0 \cdot \sin\left(\frac{\Phi}{2}\right) \mathbf{j} + 0 \cdot \sin\left(\frac{\Phi}{2}\right) \mathbf{k} \\ &= \cos\left(\frac{\Phi}{2}\right) + \sin\left(\frac{\Phi}{2}\right) \mathbf{i} \end{aligned}$$

According to eqn. (5.22), the total quaternion  $Z_D$  is then given as a function of the Euler angles:

$$Z_{D} = Z_{\Psi} \cdot Z_{\Theta} \cdot Z_{\Phi}$$

$$= \left(\cos\left(\frac{\Psi}{2}\right) + \sin\left(\frac{\Psi}{2}\right) \mathbf{k}\right) \left(\cos\left(\frac{\Theta}{2}\right) + \sin\left(\frac{\Theta}{2}\right) \mathbf{j}\right) \left(\cos\left(\frac{\Phi}{2}\right) + \sin\left(\frac{\Phi}{2}\right) \mathbf{i}\right)$$

$$= \cos\left(\frac{\Psi}{2}\right) \cos\left(\frac{\Theta}{2}\right) \cos\left(\frac{\Phi}{2}\right) + \sin\left(\frac{\Psi}{2}\right) \sin\left(\frac{\Theta}{2}\right) \sin\left(\frac{\Phi}{2}\right)$$

$$+ \left(\cos\left(\frac{\Psi}{2}\right) \cos\left(\frac{\Theta}{2}\right) \sin\left(\frac{\Phi}{2}\right) - \sin\left(\frac{\Psi}{2}\right) \sin\left(\frac{\Theta}{2}\right) \cos\left(\frac{\Phi}{2}\right)\right) \mathbf{i}$$

$$+ \left(\cos\left(\frac{\Psi}{2}\right) \sin\left(\frac{\Theta}{2}\right) \cos\left(\frac{\Phi}{2}\right) + \sin\left(\frac{\Psi}{2}\right) \cos\left(\frac{\Theta}{2}\right) \sin\left(\frac{\Phi}{2}\right)\right) \mathbf{j}$$

$$+ \left(\sin\left(\frac{\Psi}{2}\right) \cos\left(\frac{\Theta}{2}\right) \cos\left(\frac{\Phi}{2}\right) - \cos\left(\frac{\Psi}{2}\right) \sin\left(\frac{\Theta}{2}\right) \sin\left(\frac{\Phi}{2}\right)\right) \mathbf{k}$$
(5.23)

### 5.6.8 Calculation of the Quaternion from the Transformation Matrix

The pragmatic way to calculate the quaternion from the transformation matrix with the tools presented so far would first determine the Euler angles from the transformation matrix and then the quaternion from the Euler angles. However, since the known problems with the Euler angles occur (computationally time-intensive trigonometric functions, "gimbal lock", c.f. secrefQuaternionen), a direct alternative is explained below.

Eqn. (5.18) represents the transformation matrix  $M_{fg}$  as a function of the quaternion components a, b, c, and d. To calculate the first quaternion component a, the main diagonal elements of the transformation matrix are summed up:

$$M_{11} + M_{22} + M_{33} = \left(a^2 + b^2 - c^2 - d^2\right) + \left(a^2 - b^2 + c^2 - d^2\right) + \left(a^2 - b^2 - c^2 + d^2\right)$$
  
=  $3a^2 - b^2 - c^2 - d^2$  (5.24)

The condition that the quaternion has a magnitude of unity:

$$a^2 + b^2 + c^2 + d^2 = 1$$

Can be solved for  $b^2$ 

$$b^2 = 1 - a^2 - c^2 - d^2$$

and substituted in eqn. (5.24)

$$M_{11} + M_{22} + M_{33} = 3a^2 - \left(1 - a^2 - c^2 - d^2\right) - c^2 - d^2 = 4a^2 - 1$$
 (5.25)

So that eqn. (5.25) can be solved for the sought quaternion component:

$$a = \frac{\sqrt{M_{11} + M_{22} + M_{33} + 1}}{2} \tag{5.26}$$

Two matrix elements can be carefully chosen, such that:

$$M_{23} - M_{32} = 2(cd + ab) - 2(cd - ab) = 4ab$$

each provides a determining equation for the remaining quaternion components:

$$b = \frac{M_{23} - M_{32}}{4a} \qquad c = \frac{M_{31} - M_{13}}{4a} \qquad d = \frac{M_{12} - M_{21}}{4a} \tag{5.27}$$

Unfortunately, when calculating the quaternion according to eqs (5.26) - (5.27), there is the problem that the quaternion component a – which is present in eqn. (5.27) in all denominators – can become zero, so that the other components can no longer be calculated. The component a, which – according to eqn. (5.23) – results from

$$a = \cos\left(\frac{\Psi}{2}\right)\cos\left(\frac{\Theta}{2}\right)\cos\left(\frac{\Phi}{2}\right) + \sin\left(\frac{\Psi}{2}\right)\sin\left(\frac{\Theta}{2}\right)\sin\left(\frac{\Phi}{2}\right)$$

disappears, for example, if one Euler angle has the value 0 and another the value  $\pi$ , since then both a cosine and a sine and thus both summands become zero. In this case, the calculation can begin with the second quaternion component b by summing the diagonal elements of the transformation matrix with different signs:

$$M_{11} - M_{22} - M_{33} = \left(a^2 + b^2 - c^2 - d^2\right) - \left(a^2 - b^2 + c^2 - d^2\right) - \left(a^2 - b^2 - c^2 + d^2\right)$$
$$= -a^2 + 3b^2 - c^2 - d^2$$

so that, using

$$a^2 = 1 - b^2 - c^2 - d^2$$

yields a definition for b:

$$M_{11} - M_{22} - M_{33} = -\left(1 - b^2 - c^2 - d^2\right) + 3b^2 - c^2 - d^2 = 4b^2 - 1$$
  
$$\Rightarrow \quad b = \frac{\sqrt{M_{11} - M_{22} - M_{33} + 1}}{2}$$

With b, the other components can then also be determined:

$$c = \frac{M_{12} + M_{21}}{4b}$$
  $d = \frac{M_{13} + M_{31}}{4b}$   $a = \frac{M_{23} - M_{32}}{4b}$ 

In total, there are four sets of determining equations, depending on which diagonal element signs and thus which quaternion component is started with:

$$\begin{aligned} a &= \frac{\sqrt{M_{11} + M_{22} + M_{33} + 1}}{2} & b = \frac{M_{23} - M_{32}}{4a} & c = \frac{M_{31} - M_{13}}{4a} & d = \frac{M_{12} - M_{21}}{4a} \\ b &= \frac{\sqrt{M_{11} - M_{22} - M_{33} + 1}}{2} & c = \frac{M_{12} + M_{21}}{4b} & d = \frac{M_{13} + M_{31}}{4b} & a = \frac{M_{23} - M_{32}}{4b} \\ c &= \frac{\sqrt{-M_{11} + M_{22} - M_{33} + 1}}{2} & d = \frac{M_{23} + M_{32}}{4c} & a = \frac{M_{31} - M_{13}}{4c} & b = \frac{M_{12} + M_{21}}{4c} \\ d &= \frac{\sqrt{-M_{11} - M_{22} + M_{33} + 1}}{2} & a = \frac{M_{12} - M_{21}}{4d} & b = \frac{M_{13} + M_{31}}{4d} & c = \frac{M_{23} + M_{32}}{4d} \end{aligned}$$

Since the quaternion to be calculated is a unit quaternion, at least one of its components must be significantly different from zero so that it can be started with. In numerical practice, one can simply search for the largest radicant  $\pm M_{11} \pm M_{22} \pm M_{33} + 1$  to decide which set of governing equations to use.

#### 5.6.9 Differential Equation of the Quaternions

Under the important assumption that the quaternion Z = a + bi + cj + dk is a unit quaternion, the attitude differential eqn. (5.1) can be replaced by the very compact quaternion differential equation:

$$\dot{Z} = \frac{1}{2} \cdot Z \cdot Z_{\Omega} \tag{5.28}$$

Here  $Z_{\Omega}$  is a pure quaternion (with vanishing real part), whose imaginary part consists of the three elements  $p_{Kf}$ ,  $q_{Kf}$ , and  $r_{Kf}$  of the body-fixed angular velocity vector  $\Omega_{Kf}$ :

$$Z_{\Omega} = 0 + p_{Kf}\mathbf{i} + q_{Kf}\mathbf{j} + r_{Kf}\mathbf{k}$$

On the right-hand side of the quaternion differential eqn. (5.28), the quaternion product can be multiplied out:

$$\dot{Z} = \frac{1}{2} \cdot Z \cdot Z_{\Omega}$$

$$= \frac{1}{2} \cdot (a + b\mathbf{i} + c\mathbf{j} + dk) \cdot (0 + p_{Kf}\mathbf{i} + q_{Kf}\mathbf{j} + r_{Kf}\mathbf{k})$$

$$= \frac{1}{2} \cdot \{ (-p_{Kf} \cdot b - q_{Kf} \cdot c - r_{Kf} \cdot d)$$

$$+ (p_{Kf} \cdot a + r_{Kf} \cdot c - q_{Kf} \cdot d)\mathbf{i}$$

$$+ (q_{Kf} \cdot a - r_{Kf} \cdot b + p_{Kf} \cdot d)\mathbf{j}$$

$$+ (r_{Kf} \cdot a + q_{Kf} \cdot b - p_{Kf} \cdot c)\mathbf{k} \}$$
(5.29)

#### 5.6.10 Numerical Simulation

Often – for example for numerical simulation – eqn. (5.29) is represented in matrix notation by expressing the quaternion Z = a + bi + cj + dk in terms of a real column vector  $\mathbf{Z} = \begin{bmatrix} a \ b \ c \ d \end{bmatrix}^T$  and the right-hand side of eqn. (5.29) as a matrix-vector product:

$$\dot{\boldsymbol{Z}} = \begin{bmatrix} \dot{a} \\ \dot{b} \\ \dot{c} \\ \dot{d} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -p_{Kf} & -q_{Kf} & -r_{Kf} \\ p_{Kf} & 0 & r_{Kf} & -q_{Kf} \\ q_{Kf} & -r_{Kf} & 0 & p_{Kf} \\ r_{Kf} & q_{Kf} & -p_{Kf} & 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \frac{1}{2} \cdot \boldsymbol{M}_{\boldsymbol{\Omega}} \cdot \boldsymbol{Z}$$
(5.30)

During a longer simulation, unavoidable numerical errors can lead to the prerequisite for eqn. (5.30), namely the fact that Z is a unit quaternion, no longer being fulfilled.

In order to keep this error as small as possible, a simple proportional control is suitable. The control error  $\Delta Z = 1 - |\mathbf{Z}|$ , i. e. the deviation of the quaternion's magnitude from one, is thereby scaled with the current state  $\mathbf{Z}$  and fed back to all components of the quaternion integrator via a proportional controller with the gain K. The extended eqn. (5.30) then reads:

$$\dot{\boldsymbol{Z}} = \frac{1}{2} \cdot \boldsymbol{M}_{\boldsymbol{\Omega}} \cdot \boldsymbol{Z} + K \cdot \Delta \boldsymbol{Z} \cdot \boldsymbol{Z}$$

$$= \frac{1}{2} \cdot \boldsymbol{M}_{\boldsymbol{\Omega}} \cdot \boldsymbol{Z} + K \left(1 - |\boldsymbol{Z}|\right) \boldsymbol{Z}$$

$$= \frac{1}{2} \cdot \boldsymbol{M}_{\boldsymbol{\Omega}} \cdot \boldsymbol{Z} + K \left(1 - \sqrt{a^2 + b^2 + c^2 + d^2}\right) \boldsymbol{Z}$$
(5.31)

To save computing time, the magnitude in eqn. (5.31) can be replaced by the magnitude squared without qualitatively changing the control behaviour:

$$\dot{\boldsymbol{Z}} = \frac{1}{2} \cdot \boldsymbol{M}_{\boldsymbol{\Omega}} \cdot \boldsymbol{Z} + K \left( 1 - \left( a^2 + b^2 + c^2 + d^2 \right) \right) \boldsymbol{Z}$$
(5.32)

As with any controller design, a compromise has to be found for the controller gain K. If K is too small, the quaternion amount can deviate significantly from one, while a too large K leads to a stiffening of the system to be simulated, with the consequence of longer computation times and the danger of numerical instability.

The presented method for preserving the quaternion unit length has the advantage that it is not necessary to directly change the quaternion – which is part of the state vector of the simulation. However, if the simulation environment allows the direct setting of state variables without major effort, the quaternion can simply be made into a unit quaternion again after each integration step by dividing it by its magnitude.

# Chapter 6

# Controller Design

## 6.1 Natural Response

## 6.1.1 Division of the State Variables into Longitudinal and Lateral Motion

	$arOmega_K$	$V_K$	${\it \Phi}$	8
Longitudinal motion	$q_K$	$u_K  w_K$	Θ	x $z$
Lateral motion	$p_K r_K$	$v_K$	$\phi$ $\psi$	y

## 6.1.2 Longitudinal Motion



Figure 6.1: Differential equations and output equations of longitudinal motion (without wind).



Figure 6.2: Pole distribution of the longitudinal motion.





- " $\alpha$  oscillation"
- pitch oscillation in  $q_K$ ,  $\alpha$ ,  $\Theta$
- high frequency (e.g.:  $f = 0.1 \,\text{Hz}, T = 10 \,\text{s}$ )
- medium damping (e. g.: D = 0.5)





Figure 6.4: Phygoid oscillation.

- "energy oscillation"
- flight-path oscillation in  $V_k, \gamma$
- low frequency (e.g.: f = 0.01 Hz, T = 100 s)
- can become unstable (e.g.: D = 0)

## 6.1.3 Lateral Motion



Figure 6.5: Differential equations and output equations of lateral motion (without wind).



Figure 6.6: Pole distribution of the lateral motion.





- angular oscillation in  $\beta$ ,  $\Psi$ ,  $\Phi$ ,  $p_k$ ,  $r_k$
- high frequency (e.g.:  $f = 0.1 \,\text{Hz}, T = 10 \,\text{s}$ )
- low damping (e.g.: D = 0.1)

#### 6.1.3.2 Roll motion

- "roll low pass", "roll delay"
- aperiodic roll motion in  $p_k, \Phi$
- time constant: e.g.: T = 1 s

- "open integrator" with "some" feedback
- $\xi$  block  $\rightarrow$  stationary bank angle
- can be unstable  $\rightarrow$  spiral dive

## 6.2 Trim Calculation

Trimming is the calculation of trim variables (input and state variables) so that specified trim requirements (output and state derivative variables) are met. The trim calculation determines the (mostly steady-state) initial state of a simulation.

#### 6.2.1 Horizontal Straight Flight

Task: An un-accelerated, slip-free, horizontal straight flight without wind is to be trimmed. The relevant input, state and output variables of the longitudinal motion are:

Input variables:

$$\boldsymbol{u} = \begin{bmatrix} F & \eta \end{bmatrix}^T$$

State variables:

$$\boldsymbol{x} = \begin{bmatrix} q_{Kf} & u_{Kf} & w_{Kf} & \Theta \end{bmatrix}^T$$

Output variables:

$$\boldsymbol{v} = \begin{bmatrix} V_K & \gamma & \alpha & \cdots \end{bmatrix}^T$$

- Firs Idea: Horizontal flight is defined by the specification of a certain flight-path velocity  $V_K$  and by the requirement of a vanishing angle of climb  $\gamma = 0$ . These two trim requirements can be fulfilled by the two trim variables thrust F and elevator  $\eta$ . Thrust and elevator adjust both the energy and the moment balance of the longitudinal motion in such a way that an unaccelerated flight at a constant altitude is possible.
- Second Idea: Unfortunately, from a physical point of view, thrust and elevator do not directly influence the desired flight-path speed and angle of climb. Thrust and elevator generate forces and moments that directly lead only to accelerations and angular accelerations. The (angular) accelerations are then integrated by the (angular) velocity integrators of the kinetics into velocities and angular velocities. In the second integration step, position and attitude follow from this.

For example, the elevator  $\eta$  essentially generates a pitch moment M, which leads directly to a pitch acceleration  $\dot{q}_K$  via the pitch moment of inertia. The first integration then turns the pitch acceleration into a pitch speed  $q_K$  and the second integrator generates the pitch angle  $\Theta$  from this. Pitching simultaneously changes the angle of attack  $\alpha$ , which leads to a change of the lift A. The changed lift (vertical force) then generates a vertical acceleration  $\dot{w}_K$ , which in turn is integrated into a vertical speed  $w_K$ . Only this vertical velocity results in a vertical position change and thus the desired change of the angle  $\gamma$ .

Brielfy: A trim algorithm cannot directly set the desired trim requirement (angle of climb) by "tweaking" the elevator trim value.

**Third Idea:** The internal state variables involved in the definition of the desired flight state  $(q_{Kf}, u_{Kf}, w_{Kf}, \Theta)$  must also be trimmed. The determination of the pitch speed  $q_{Kf}$  is simple. It must disappear for a steady-state straight flight, as otherwise the aircraft would pitch up or down permanently.

To determine the remaining three trim values  $(u_{Kf}, w_{Kf}, \Theta)$ , three more trim requirements must be found. These follow directly from the requirement for an unaccelerated flight: neither pitch accelerations nor translational accelerations may occur:  $\dot{q}_{Kf} = \dot{u}_{Kf} = \dot{w}_{Kf} = 0$ .

If the flight condition is trimmed out, all other output variables follow automatically: the angle of attack  $\alpha$ , for example, is calculated directly from  $\alpha = \Theta - \gamma$  for pure longitudinal motion without wind.

#### 6.2.2 Generalisation

The knowledge gained from the example of the longitudinal motion of an aircraft can be generalised. A general non-linear dynamic system can be described by a vector differential equation and an algebraic vector output equation:

Vector differential equation:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$$

Vector output equation:

$$\boldsymbol{v} = \boldsymbol{g}\left(\boldsymbol{x}, \, \boldsymbol{u}\right)$$



Figure 6.8: General non-linear dynamic system.

Trim requirements (known)Elements of  $\dot{\boldsymbol{x}}$  and  $\boldsymbol{v}$  (left side of equations)Trim variables (unknown)Elements of  $\boldsymbol{x}$  and  $\boldsymbol{u}$  (right side of equations)

#### 6.2.2.1 Procedure

1. For each trim requirement, leave a trim variable free that can fulfil (or at least influence) the requirement.

- 2. For each trim variable, establish a trim requirement by which the trim variable is defined (or at least constrained).
- 3. Set the elements of  $\boldsymbol{x}$  and  $\boldsymbol{u}$ , that are not trim variables to fixed values.
- 4. The elements of  $\dot{x}$  and v, that are not trim requirements follow automatically.

## 6.2.3 Engine dynamics

Each dynamic subsystem (actuator dynamics, sensors, filters, controllers,  $\ldots$ ) of the overall system to be trimmed must also be trimmed.

If, for example, an engine is modelled as a limited first-order system and the thrust command  $F_c$  is searched for, which causes a desired flight-path speed  $V_K$ , then the output F of the thrust integrator must be seen as an additional trim variable and its input  $\dot{F}$  as a further trim requirement. Of course, when trimmed out, the thrust F will be equal to the thrust command  $F_c$ ; however, both are unknown and must therefore be determined together by the trim program.

Each additional trim variable requires exactly one additional trim requirement. Therefore, for a stationary trim point (constant thrust), the derivative of the thrust is required to be zero:  $\dot{F} = 0$ .



Figure 6.9: Engine dynamics with saturation.

#### 6.2.3.1 Saturation

Saturation is a serious problem for many trimming algorithms. Usually, the optimiser in the trim program varies the trim variables according to a more or less intelligent procedure until the objective function containing the trim requirements has become better than a given limit. If, however, during this search procedure a saturation is triggered because the trim algorithm has increased a trim variable above its maximum value "on a trial basis", suddenly a small variation of the trim variable no longer causes any change in the trim requirements.

The gradient-oriented trim algorithm "no longer knows in which direction it should continue to optimise" and aborts with an error message.

In these cases, it often helps to reduce the maximum step size of the corresponding trim variable so that the algorithm does not "accidentally" violate the saturation in a step that is too large. In addition, it is of course useful to set each trim variable to an estimated initial value that is as close as possible to the expected trim point. For example, if the maximum thrust of an engine in cruise flight is 90 kN, it certainly makes more sense to set the initial value of the trim variable thrust to perhaps 60 kN rather than to 0 kN or even 100 kN.

## 6.3 Basic Controller

The basic controller dampens the natural response (pitch, yaw and roll dampers) and controls the airspeed, pitch angle, bank angle and side-slip angle. The commands for these input variables of the basic controller can either come directly from the pilot (e.g. sidestick commanded *rate command attitude hold* for pitch and roll angle) or from a superimposed flight-path control loop.



## 6.3.1 Basic Controller of Longitudinal Motion

Figure 6.10: Basic controller of the longitudinal motion.

- $K_{\eta q}$  The **pitch damper** uses the elevator to dampen the short period. In doing so, the (measured) pitch speed is compared with a reference value of zero (so to speak), and the elevator is always deflected in such a way that the resulting pitch moment and the resulting pitch acceleration counteract the pitch speed.
- $K_{\Theta}$  The **pitch controller** uses the same control input (elevator) as the pitch damper and controls the pitch angle. The pitch angle reference can be set directly by the pilot (e. g. by means of the longitudinal sidestick) or by an outer (cascaded) flightpath controller (altitude controller, ...).
- $K_V$  The **airspeed controller** (autothrottle) uses the thrust to maintain an airspeed reference. To prevent a steady-state control error, the airspeed controller can have an integral component.

## 6.3.2 Basic Controller of the Lateral Motion



Figure 6.11: Basic controller of lateral motion.

- $K_{\zeta r}$  The **yaw damper** measures the yaw rate and uses the rudder to dampen the Dutch roll. During a stationary turn, the yaw damper should not be active, if possible, so as not to suppress the yaw rate that is then desired ( $\rightarrow$  use of a high pass, which does not "let through" constant yaw rates).
- $K_{\xi p}$  The **roll damper** uses the aileron to change the roll time constant.
- $K_{\beta}$  The side-slip angle controller uses the rudder to control the side-slip angle. Often, for example in a coordinated turn, the side-slip angle should disappear (reference value equal to zero) to ensure a symmetrical, economical flow. The side-slip angle reference can be commanded with the pedals, for example, or it is specified by the autopilot (e. g. during the decrab manoeuvre to align the landing gear in the direction of the runway when approaching with a crosswind).
- $K_{\Phi}$  The **bank angle controller** uses the same control input (aileron) as the roll damper and provides for the control of the bank angle. The desired bank angle can be set directly by the pilot (e. g. by means of the lateral sidestick) or by an outer (cascaded) flight-path controller controller (course controller, ...).

## 6.4 Flight-Path Controller

The flight-path controller (altitude controller and flight-path azimuth controller) uses the "basic-controlled" aircraft as a "modified plant" in the sense of a cascade control. The control variables of the basic controller (pitch angle and bank angle) are directly commanded by the flight-path controller.

## 6.4.1 Cascade Control



Figure 6.12: Cascade control.

- Outer control loop supplies reference for inner control loop.
- Design the inner control loop to be fast and without steady-state accuracy.
- Outer control loop provides steady-state accuracy.

### 6.4.2 Altitude and Flight-Path Azimuth Control



Figure 6.13: Altitude and flight-path azimuth control.

- $K_H$  In this cascade control, the **altitude controller** does not have direct access to the elevator, but commands a pitch angle reference to the basic controller of the longitudinal motion, which in turn intervenes in the aircraft's moment balance via the elevator in order to adjust the pitch angle accordingly. In order to achieve steady-state accuracy for altitude control, the altitude controller can be designed as a PI(D) controller.
- $K_{\chi}$  The **flight-path azimuth controller** controls the flight-path azimuth by passing on a reference for the bank angle to the basic controller. As an alternative to the flight-path azimuth, the yaw angle (heading) can be used as a controlled variable (e.g. for availability reasons).

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