Automatic Control and Flight Control

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Contents

Nomenclature

α	angle of attack
α	subjective weighting factor
$\bar{q}$	dynamic pressure
β	sideslip angle
χ	flight-path azimuth
$\delta(t)$	ideal impulse (Dirac impulse)
δ_η	stick deflection (pull)
δ_ξ	stick deflection (to the right)
δ_ζ	pedal deflection
η	elevator deflection
γ	flight-path (inclination) angle, angle of climb
$\Im(z)$	imaginary part of a complex number
$\mathcal{L}$	Laplace operator
$\mathcal{L}^{-1}$	inverse Laplace operator
∇	Nabla operator
ω	angular frequency
ω_0	natural angular frequency
ω_E	corner frequency
ω_m	center frequency
$\overline{Z}$	conjugate quaternion
$\Re(z)$	real part of a complex number
ρ	air density
τ	time (as integration variable)

ε	angle of the pole with respect to the imaginary axis
ε	infinitesimal small value
$\mathit{\Phi}$	roll angle, bank angle
φ	phase angle, phase shift
$\mathit{\Psi}$	yaw angle
$\mathit{\Theta}$	pitch angle
$\boldsymbol{\Omega}$	angular velocity vector
$\boldsymbol{\mathit{\Phi}}$	attitude vector (Euler angle vector)
$\boldsymbol{A}$	system matrix, state matrix
$\boldsymbol{B}$	input matrix, control matrix
$\boldsymbol{C}$	output matrix, measurement matrix
$\boldsymbol{D}$	angular momentum vector
$\boldsymbol{D}$	feedforward matrix, direct transmission matrix
$\boldsymbol{f}$	vector differential equation
$\boldsymbol{G}$	weight vector
$\boldsymbol{g}$	vector output equation
$\boldsymbol{I}$	inertia tensor
$\boldsymbol{M_{fa}}$	transformation matrix from the aerodynamic to the body-fixed axis system
$\boldsymbol{M_{fg}}$	transformation matrix from the geodetic to the body-fixed axis system
$\boldsymbol{M_{kg}}$	transformation matrix from the geodetic to the flight-path axis system
$\boldsymbol{M}_{\Omega}$	quaternion differential equation matrix
$\boldsymbol{n}$	axis of rotation vector of a quaternion
$\boldsymbol{P}$	momentum vector
$\boldsymbol{Q}$	moment vector
$\boldsymbol{r_F}$	distance of the engine from the reference point
$\boldsymbol{R}$	resulting force vector
$\boldsymbol{s}$	position vector (direction, distance)
$\boldsymbol{u}$	input vector
$\boldsymbol{V}$	velocity vector

$\boldsymbol{v}$	output vector
$\boldsymbol{x}$	state vector
Ξ	angle of rotation of a quaternion
ξ	aileron deflection
ζ	rudder deflection
A	amplitude (ratio)
A	area
A	lift
a	real part of a quaternion
b, c, d	imaginary parts of a quaternion
c	spring constant
C_A	lift coefficient
C_l	roll moment coefficient
C_m	pitch moment coefficient
C_n	yaw moment coefficient
C_Q	side force coefficient
C_W	drag coefficient
$C_{A\alpha}$	lift due to angle of attack etc.
C_{W0}	zero drag
D	damping
$d\,(t)$	square wave pulse
dB	dezibel
E	aerodynamic force unit
$e\,(t)$	control error, control difference, deviation
e_{max}	maximum overshoot
F	force
f	frequency
$f\,(t)$	ramp response in time domain
F_c	thrust reference

g	gravitational acceleration
$G(s)$	transfer function
$g(t)$	impulse response in time domain
$G(z)$	transfer function in the z-domain
G_0	open loop transfer function
G_g	overall transfer function
G_M	sensor transfer function
G_R	controller transfer function
G_S	plant transfer function
G_V	forward transfer function
G_V	open loop controller transfer function
G_z	disturbance transfer function
G_{ew}	reference control error
G_{ez}	disturbance control error
G_{St}	disturbance feedforward transfer function
$H(s)$	step response in frequency domain
$h(t)$	step response in time domain
I	integral criteria
i, j, k	imaginary units of a quaternion (or of a complex number)
I_x, I_y, I_z	moments of inertia
I_{xz}	product of inertia
K	gain
k	integer number
K_β	sideslip angle controller
K_χ	flight-path azimuth controller
K_Φ	bank angle controller
K_Θ	pitch controller
K_H	altitude controller
K_H	auxiliary variable

K_R	controller gain
K_V	airspeed controller, autothrottle
$K_{\eta q}$	pitch damper
$K_{\xi p}$	roll damper
$K_{\zeta r}$	yaw damper
$K_{R_{krit}}$	amplitude reserve, critical gain
L	roll moment
l_μ	mean aerodynamic chord
M	pitch moment
m	mass
Ma	Mach number
N	yaw moment
p	roll speed
p_A^*	normalised aerodynamic roll speed etc.
Q	side force
q	pitch speed
r	damping factor
r	magnitude of a complex number
r	yaw speed
$r(t)$	unit ramp
s	Laplace variable
$S(s)$	unit step in frequency domain
$s(t)$	unit step in time domain
T	period
T	sampling time
t	time
t_ε	magnitude of the control error stays less than ε
T_D	D-element time constant
T_F	engine time constant

T_g	compensation time
T_I	integrator time constant
T_N	normalisation time constant
T_N	reset time
T_T	dead time constant
T_u	delay time
T_V	reset time
t_{an}	response time
T_{krit}	period of the critical oscillation
u	forward speed
$U(s)$	input variable in frequency domain
$u(t)$	input variable in time domain
u_k	input variable in z-domain
v	side speed
$V(s)$	output variable in frequency domain
$v(t)$	output variable in time domain
V_A	aerodynamic speed, air speed
v_k	output variable in z-domain
W	drag
w	sink speed
$w(t)$	reference input variable
X	x component of the force vector (towards the front)
x	state variable
x	x component of the position vector (towards the front)
$x(t)$	controlled variable
Y	y component of the force vector (towards the right)
y	y component of the position vector (towards the right)
$y(t)$	manipulated variable
Z	quaternion

Z	z component of the force vector (towards the bottom)
z	complex number
z	independent variable of the transfer function in the z-domain
z	z component of the position vector (towards the bottom)
$z\left(t\right)$	disturbance
Z^0	unit quaternion
Z_D	unit quaternion of rotation
Z_Ω	flight-path angular velocity quaternion
Index A	aerodynamic
Index a	aerodynamic axis system
Index F	thrust
Index f	body-fixed axis system
Index g	geodetic axis system, earth-fixed axis system
Index K	flight-path
Index k	flight-path axis system
Index W	wind
PFD	partial fraction decomposition

Part I

Basics of automatic control

Chapter 1

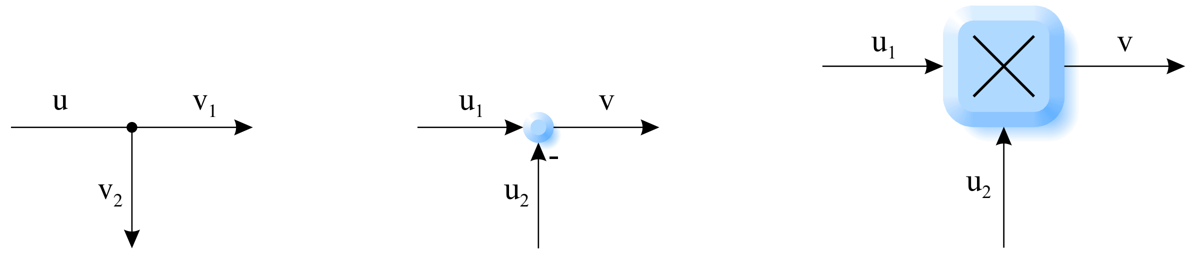
Introduction

1.1 Block diagrams and terminology

1.1.1 Norms

DIN 19221	Regelungstechnik und Steuerungstechnik
DIN 19225	Benennung und Einteilung von Reglern
DIN 19226	Regelungstechnik und Steuerungstechnik
VDI/VDE 3526	Benennungen für Steuer- und Regelschaltungen

1.1.2 Signal connections



(a) Branch point: $v_1 = v_2 = u$

(b) Addition or subtraction point: $v = u_1 - u_2$

(c) Multiplication point: $v = u_1 \cdot u_2$

Figure 1.1: Signal connections

1.1.3 Blocks

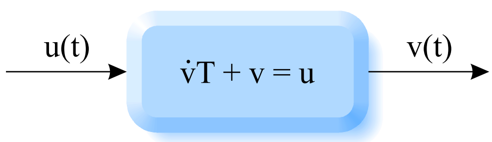


Figure 1.2: Differential equation

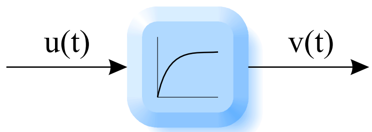


Figure 1.3: Step response

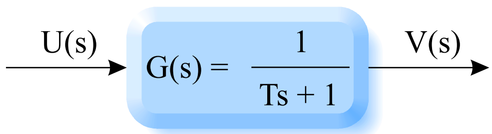


Figure 1.4: Transfer function

1.1.4 Terminology

u	input variable
v	output variable
w	reference input variable, command, setpoint variable
x	control(led) variable
e	control error, control difference, deviation, error signal,
y	manipulated variable, correcting variable
z	disturbance (variable)
s	Laplace variable, complex frequency
$G(s)$	transfer function

Table 1.1: Variables of automatic control

1.2 Open loop and closed loop control

1.2.1 Open loop control

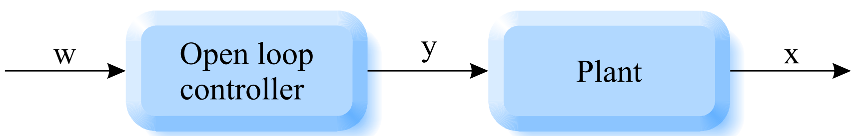


Figure 1.5: Open loop control

- open chain of effects, no feedback

- can only compensate for known disturbances
- cannot become unstable
- fast

1.2.2 Closed loop control

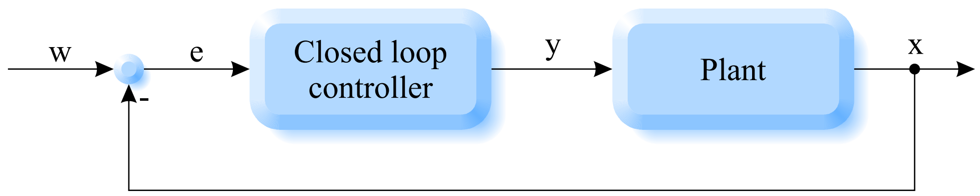


Figure 1.6: Closed loop control

- closed control loop
- can also compensate for unknown disturbances
- may become unstable
- slower

1.3 Special excitation functions and system responses

1.3.1 Step

$$s(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0.5 & \text{if } t = 0 \\ 0 & \text{if } t < 0 \end{cases}$$

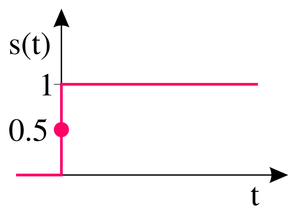


Figure 1.7: Unit step

1.3.2 Ramp

$$r(t) = \begin{cases} t & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

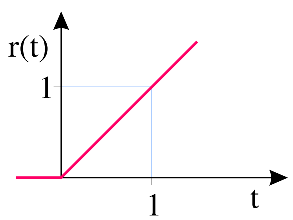


Figure 1.8: Unit ramp

1.3.3 Square wave pulse

$$d\left(t\right)=\begin{cases}1/\varepsilon & \text{if } 0\leq t\leq\varepsilon\\ 0 & \text{otherwise}\end{cases}$$

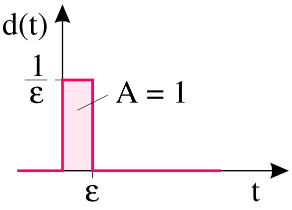


Figure 1.9: Unit square wave pulse

1.3.4 Ideal impulse (Dirac impulse)

$$\delta\left(t\right)=\lim_{\varepsilon\to0}d\left(t\right)=\begin{cases}\infty & \text{if } t=0\\ 0 & \text{otherwise}\end{cases}$$

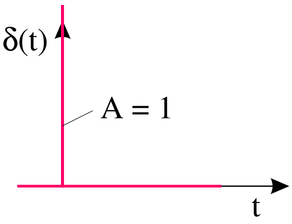


Figure 1.10: Ideal impulse (Dirac impulse)

1.3.5 Sine

$$x\left(t\right)=\sin\omega t$$

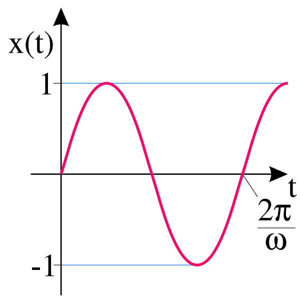


Figure 1.11: Unit sine

1.3.6 Relation between ramp, step and impulse

$$r\left(t\right)\ \underset{\int \mathrm{d}t}{\overset{\frac{\partial}{\partial t}}{\rightleftarrows}}\ s\left(t\right)\ \underset{\int \mathrm{d}t}{\overset{\frac{\partial}{\partial t}}{\rightleftarrows}}\ \delta\left(t\right)$$

1.3.7 System responses

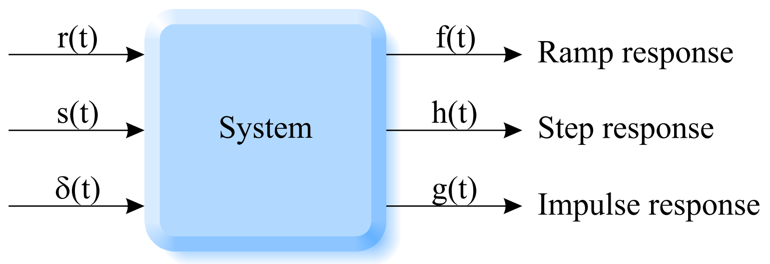


Figure 1.12: System responses

1.4 Static and dynamic behaviour

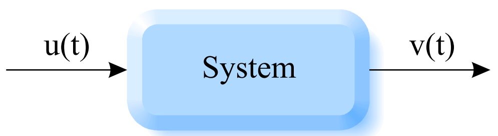


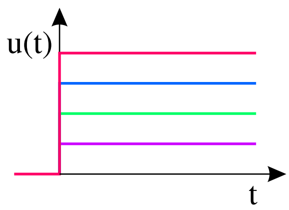
Figure 1.13: General (static or dynamic) system

1.4.1 Static behaviour (static characteristic curve)

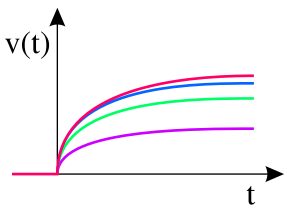
- example: ideal measuring amplifier
- change in the output variable only if the input variable is currently changing
- no energy storage $\rightarrow$ no self-movement (no intrinsic dynamics)
- description using algebraic equation

1.4.2 Dynamic behaviour

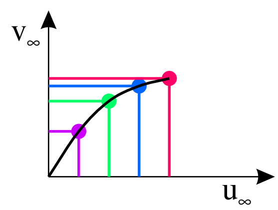
- example: pendulum
- change in the output variable without the input variable currently changing
- internal energy storage is charged and discharged $\rightarrow$ self-movement (intrinsic dynamics)
- description using differential equations



(a) Step inputs



(b) Step outputs



(c) Steady-state (static) nonlinear characteristic

Figure 1.14: Dynamic behaviour

1.5 Linear and non-linear behaviour

In many ways, linear systems are more pleasant than non-linear systems. They are generally much easier to analyze, control and simulate. To check whether a general nonlinear system of the form $v = g(u)$ is linear, two linearity conditions are used, both of which must be fulfilled:

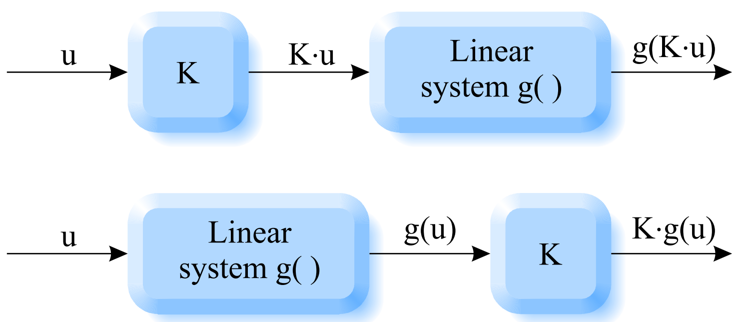


Figure 1.15: Gain principle: $g(K \cdot u) \stackrel{!}{=} K \cdot g(u)$

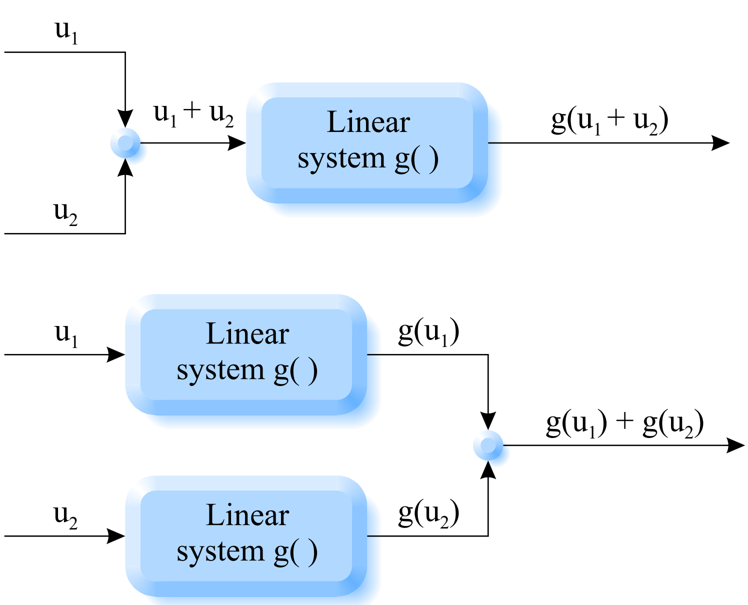


Figure 1.16: Superposition principle: $g(u_1 + u_2) \stackrel{!}{=} g(u_1) + g(u_2)$

✗ "If it works for one input amplitude, it always works."

✗ Order of linear blocks interchangeable: $g\left(h\left(u\right)\right) = h\left(g\left(u\right)\right)$

1.5.1 Linear blocks

- integrator
- differentiator
- gain (constant)
- sum
- time delay
- composite blocks: P-T_1, P-T_2, PD-T_1-filter, PID-controller, ...
- general transfer function
- linear state space representation

Chapter 2

Linear Systems

2.1 Second-order mechanical oscillator

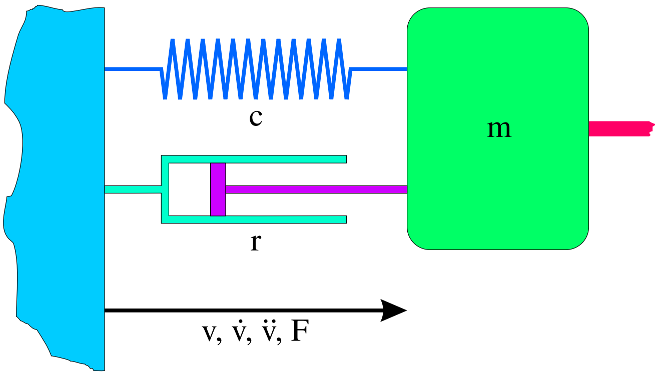


Figure 2.1: Second-order mechanical oscillator (v: displacement)

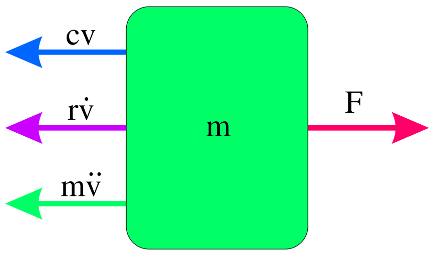


Figure 2.2: Free-body second-order oscillator

Second-order linear inhomogeneous differential equation:

$$m\ddot{v} + r\dot{v} + cv = F$$

Normal form:

$$\ddot{v} + \frac{r}{m}\dot{v} + \frac{c}{m}v = \frac{F}{m}$$

General second-order system:

$$\ddot{v} + 2D\omega_0\dot{v} + \omega_0^2 v = K\omega_0^2 u$$

Input variable:

$$u = F$$

Natural angular frequency:

$$\omega_0^2 = \frac{c}{m} \quad \Rightarrow \quad \omega_0 = \sqrt{\frac{c}{m}}$$

Damping:

$$2D\omega_0 = \frac{r}{m} \quad \Rightarrow \quad D = \frac{r}{2m\omega_0} = \frac{r}{2m\sqrt{\frac{c}{m}}} = \frac{r}{2\sqrt{cm}}$$

Gain factor:

$$K\omega_0^2 = \frac{1}{m} \quad \Rightarrow \quad K = \frac{1}{m\omega_0^2} = \frac{1}{m\frac{c}{m}} = \frac{1}{c}$$

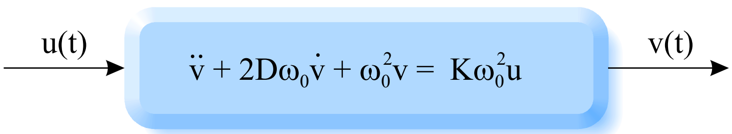
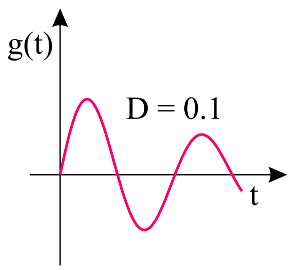


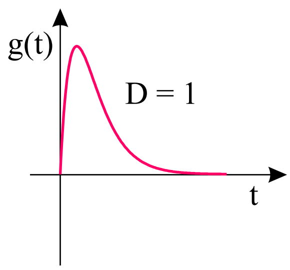
Figure 2.3: Block diagram of second-order oscillator

2.1.1 Impulse response

Impulse as input variable: $u(t) = \delta(t)$



(a) $D = 0.1$

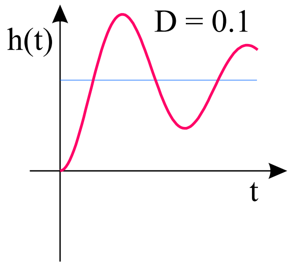


(b) $D = 1$

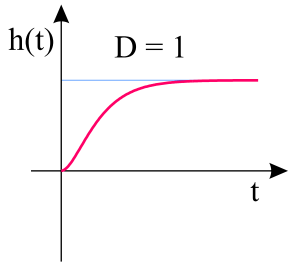
Figure 2.4: Impulse response $g(t)$

2.1.2 Step response

Step as input variable: $u\,(t) = s\,(t)$



(a) $D = 0.1$



(b) $D = 1$

Figure 2.5: Step response $h\,(t)$

2.2 Laplace transform (of a P-T$_2$)

Differential equation:

$$\ddot{v} + 2D\omega_0\dot{v} + \omega_0^2 v = K\omega_0^2 u$$

Initial values are equal to zero:

$$v(0) = \dot{v}(0) = 0$$

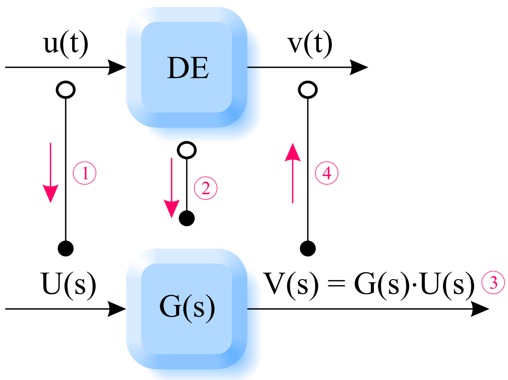


Figure 2.6: Solution of a differential equation using Laplace transform

Laplace transform:

- of the input function: $\mathcal{L}\,\{u\,(t)\} = U\,(s)$
- of the output function: $\mathcal{L}\,\{v\,(t)\} = V\,(s)$
- of the first derivative of the output function: $\mathcal{L}\,\{\dot{v}\,(t)\} = s \cdot V\,(s)$

- of the second derivative of the output function: $\mathcal{L}\left\{\ddot{v}\left(t\right)\right\} = s^2 \cdot V\left(s\right)$

Transformed differential equation:

$$s^2 V\left(s\right) + 2D\omega_0 s V\left(s\right) + \omega_0^2 V\left(s\right) = K\omega_0^2 U\left(s\right)$$

Transfer function:

$$G\left(s\right) = \frac{V\left(s\right)}{U\left(s\right)} = \frac{K\omega_0^2}{s^2 + 2D\omega_0 s + \omega_0^2}$$

2.2.1 Table of some Laplace transforms

$x\left(t\right)$	$\mathcal{L}\left\{x\left(t\right)\right\} = X\left(s\right)$
$\delta\left(t\right)$	1
$s\left(t\right) = 1$	$\frac{1}{s}$
$r\left(t\right) = t$	$\frac{1}{s^2}$
e^{-at}	$\frac{1}{s+a}$
$\mathrm{e}^{-at}\sin\omega t$	$\frac{\omega}{\left(s+a\right)^2+\omega^2}$
$\mathrm{e}^{-at}\cos\omega t$	$\frac{s+a}{\left(s+a\right)^2+\omega^2}$

Table 2.1: The most important Laplace transforms

2.2.2 Example: Step response of a P-T_2

Transfer function:

$$G\left(s\right) = \frac{1}{s^2 + s + 1} \quad \left(\omega_0 = 1,\ K = 1,\ D = 0.5\right)$$

Step:

$$\mathcal{L}\left\{s\left(t\right)\right\} = S\left(s\right) = \frac{1}{s}$$

Step response:

$$H\left(s\right) = G\left(s\right) \cdot S\left(s\right) = \frac{1}{s^2 + s + 1} \cdot \frac{1}{s}$$

Partial fraction decomposition:

$$H(s) = \frac{As+B}{s^2+s+1} + \frac{C}{s}$$

Same denominators:

$$\frac{1}{(s^2+s+1)s} = \frac{(As+B)s + C(s^2+s+1)}{(s^2+s+1)s}$$

Coefficient comparison in the numerator:

$$\begin{aligned}
s^0: &\quad 1 = C \quad \Rightarrow \quad C = 1 \\
s^1: &\quad 0 = B + C = B + 1 \quad \Rightarrow \quad B = -1 \\
s^2: &\quad 0 = A + C = A + 1 \quad \Rightarrow \quad A = -1
\end{aligned}$$

Result of the partial fraction decomposition:

$$H(s) = \frac{1}{s} - \frac{s+1}{s^2+s+1}$$

Completing the square:

$$H(s) = \frac{1}{s} - \frac{s+1}{\left(s+\frac{1}{2}\right)^2 + 1 - \frac{1}{4}} = \frac{1}{s} - \frac{s+1}{\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}}$$

Comparison with Laplace transform table (table 2.1):

$$a = \frac{1}{2} \quad \omega = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

Adapt numerator to table by splitting:

$$H(s) = \frac{1}{s} - \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}} - \frac{\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}}$$

Adapt the third addend to the table:

$$H(s) = \frac{1}{s} - \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}} - \frac{\frac{1}{2}\sqrt{\frac{3}{4}}\sqrt{\frac{4}{3}}}{\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}}$$

Ready to transform back:

$$H(s) = \frac{1}{s} - \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}} - \frac{1}{\sqrt{3}} \frac{\sqrt{\frac{3}{4}}}{\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}}$$

Step response in the time domain:

$$h\left(t\right) = \mathcal{L}^{-1}\left\{H\left(s\right)\right\} = 1 - \mathrm{e}^{-\frac{1}{2}t}\cos\sqrt{\frac{3}{4}}t - \frac{1}{\sqrt{3}}\mathrm{e}^{-\frac{1}{2}t}\sin\sqrt{\frac{3}{4}}t \tag{2.1}$$

Period:

$$\omega = \sqrt{\frac{3}{4}} = 2\pi f = \frac{2\pi}{T} \quad \Rightarrow \quad T = \frac{2\pi}{\sqrt{\frac{3}{4}}} = \frac{4\pi}{\sqrt{3}} \approx 7.2$$

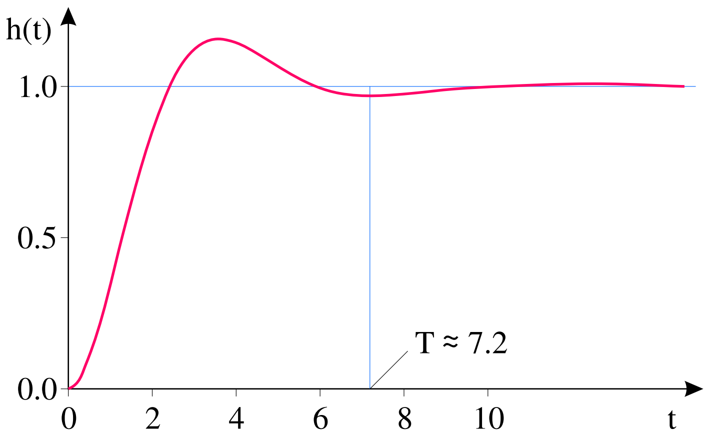


Figure 2.7: Step response of a second-order system ($D = 0.5$)

2.3 Limit theorems

Initial value theorem:

$$f\left(t = 0\right) = \lim_{t\to 0} f\left(t\right) = \lim_{s\to\infty} s \cdot F\left(s\right)$$

Final value theorem:

$$\lim_{t\to\infty} f\left(t\right) = \lim_{s\to 0} s \cdot F\left(s\right)$$

2.3.1 Example: Step response of a P-T$_2$

General (final value theorem):

$$\lim_{t\to\infty} h\left(t\right) = \lim_{s\to 0} s \cdot H\left(s\right) = \lim_{s\to 0} s \cdot G\left(s\right) \cdot \frac{1}{s} = \lim_{s\to 0} G\left(s\right)$$

Special (P-T$_2$):

$$\lim_{t\to\infty} h\left(t\right) = \lim_{s\to 0} G\left(s\right) = \lim_{s\to 0} \frac{K\omega_0^2}{s^2 + 2D\omega_0 s + \omega_0^2} = \frac{K\omega_0^2}{\omega_0^2} = K$$

2.4 Poles of the transfer function

Example: Second-order system (P-T_2)

Transfer function:

$$G(s) = \frac{K\omega_0^2}{s^2 + 2D\omega_0 s + \omega_0^2}$$

Poles are zeros of the denominator:

$$s^2 + 2D\omega_0 s + \omega_0^2 = 0$$

Two (real or complex conjugate) poles:

$$s_{1,2} = -D\omega_0 \pm \sqrt{D^2\omega_0^2 - \omega_0^2} = -D\omega_0 \pm \omega_0\sqrt{D^2 - 1}$$

Case distinction:

$\lvert D\rvert > 1$	$s_{1,2} = -D\omega_0 \pm \omega_0\sqrt{D^2-1}$	two real poles
$\lvert D\rvert < 1$	$s_{1,2} = \underbrace{-D\omega_0}_{\sigma} \pm \mathrm{j}\,\underbrace{\omega_0\sqrt{1-D^2}}_{\omega}$	complex conjugate pair of poles
$D = 0$	$s_{1,2} = \pm \mathrm{j}\omega_0$	complex conjugate pair of poles on the imaginary axis
$D = 1$	$s_{1,2} = -\omega_0$	real double pole on the left half-plane
$D = -1$	$s_{1,2} = \omega_0$	real double pole on the right half-plane (unstable)

Table 2.2: Position of the poles depending on the damping

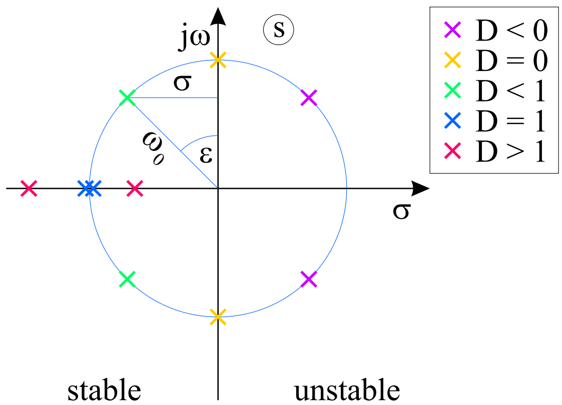


Figure 2.8: Pole distribution of a second-order system

Stability: "A system with at least one pole in the right half-plane is unstable."

Natural frequency is distance from the origin:

$$|\sigma|^2 + \omega^2 = |-D\omega_0|^2 + \left(\omega_0\sqrt{1-D^2}\right)^2 = D^2\omega_0^2 + \omega_0^2\left(1-D^2\right) = \omega_0^2$$

"The closer to the origin, the slower."

Relationship between angle ε and damping D (in the second quadrant):

$$\sin\varepsilon = \frac{|\sigma|}{\omega_0} = \frac{D\omega_0}{\omega_0} = D$$

"The closer to the imaginary axis, the worse the damping"

2.5 Frequency response

Sine input:

$$u\left(t\right) = A_u \sin\omega t$$

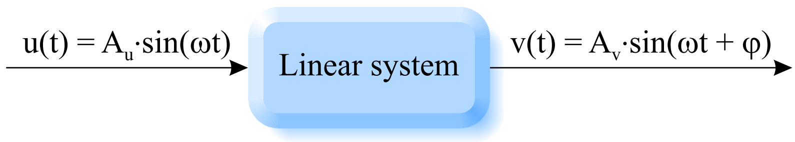


Figure 2.9: Sine input and response of a linear system

Steady-state sine response (transient process completed) has:

- same frequency ω
- different amplitude A_v
- different phase (phase shift) φ

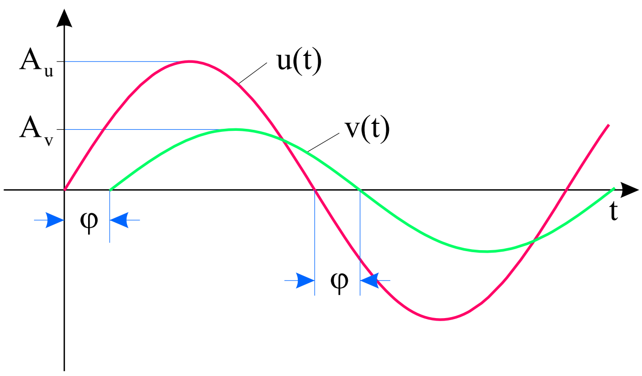


Figure 2.10: Sine input and response of a linear system

The frequency response consists of the amplitude response (ratio of output to input amplitude over frequency) and the phase response (phase shift depending on frequency):

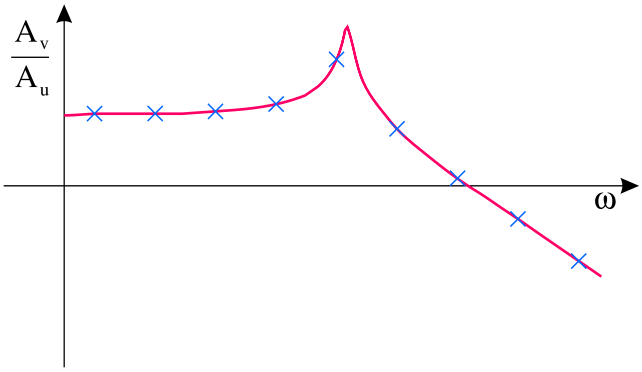


Figure 2.11: Amplitude response

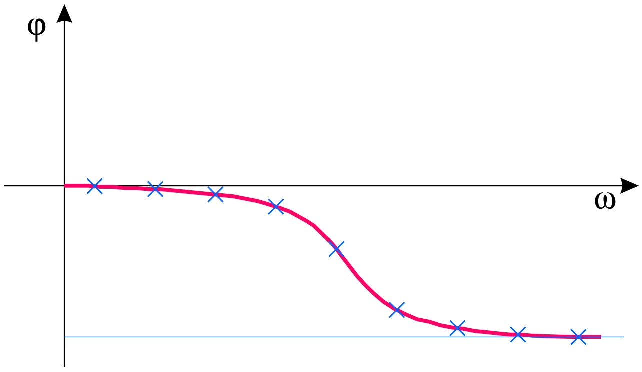


Figure 2.12: Phase response

2.6 Bode diagram and Nyquist plot

2.6.1 Properties of complex numbers

Complex number in Cartesian form:

$$z = a + \mathrm{j}b$$

Complex number in exponential form:

$$z = r \cdot e^{\mathrm{j}\varphi}$$

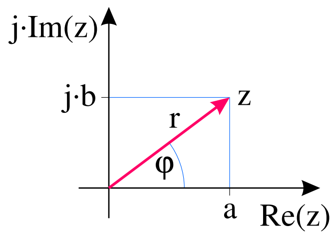


Figure 2.13: Point representation of the complex number z

Magnitude:

$$|z| = r = \sqrt{a^2 + b^2}$$

Phase:

$$\varphi = \arctan\left(\frac{b}{a}\right)$$

Real part:

$$\Re\left(z\right) = a = r \cdot \cos\left(\varphi\right)$$

Imaginary part:

$$\Im\left(z\right) = b = r \cdot \sin\left(\varphi\right)$$

Quotient of two complex numbers:

$$z_Q = \frac{r_1 \cdot \mathrm{e}^{\mathrm{j}\varphi_1}}{r_2 \cdot \mathrm{e}^{\mathrm{j}\varphi_2}} = \frac{a + \mathrm{j}b}{c + \mathrm{j}d}$$

Magnitude equals the quotient of the individual magnitudes:

$$|z_Q| = \frac{r_1}{r_2} = \frac{\sqrt{a^2 + b^2}}{\sqrt{c^2 + d^2}}$$

Phase equals the difference between the individual phases:

$$\varphi_Q = \varphi_1 - \varphi_2 = \arctan\left(\frac{b}{a}\right) - \arctan\left(\frac{d}{c}\right)$$

The real and imaginary parts result from complex conjugate expansion:

$$z_Q = \frac{a + \mathrm{j}b}{c + \mathrm{j}d} = \frac{a + \mathrm{j}b}{c + \mathrm{j}d} \cdot \frac{c - \mathrm{j}d}{c - \mathrm{j}d} = \frac{ac - a\mathrm{j}d + \mathrm{j}bc - \mathrm{j}b\mathrm{j}d}{cc - c\mathrm{j}d + \mathrm{j}dc - \mathrm{j}d\mathrm{j}d} = \frac{ac + bd + \mathrm{j}(bc - ad)}{c^2 + d^2}$$

Real part:

$$\Re\left(z_Q\right) = \frac{ac + bd}{c^2 + d^2}$$

Imaginary part:

$$\Im\left(z_Q\right) = \frac{bc - ad}{c^2 + d^2}$$

2.6.2 Logarithmic scaling (decibels)

Amplitude response	double logarithmic scaling: logarithmic frequency and amplitude in decibels
Phase response	single-logarithmic scaling: logarithmic frequency only

Converting decibels:

- $A\left[dB\right] = 20 \cdot \log_{10} A$
- $A = 10^{\frac{A[dB]}{20}}$

$A[dB]$	0	20	-40	80	6	3	$-\infty$
A	1	10	0.01	10000	≈ 2	$\approx \sqrt{2}$	0

Table 2.3: Some conversion examples

2.6.3 Example: Second-order system (P-T_2)

Transfer function:

$$G\left(s\right) = \frac{0.1}{s^2 + s + 1}$$

Frequency response:

$$\begin{aligned} G\left(\mathrm{j}\omega\right) &= \frac{0.1}{\left(\mathrm{j}\omega\right)^2 + \mathrm{j}\omega + 1} \\ &= \frac{0.1}{\left(1 - \omega^2\right) + \mathrm{j}\omega} && (2.2) \\ &= \frac{0.1}{\left(1 - \omega^2\right) + \mathrm{j}\omega} \cdot \frac{\left(1 - \omega^2\right) - \mathrm{j}\omega}{\left(1 - \omega^2\right) - \mathrm{j}\omega} \\ &= \frac{0.1 \cdot \left(1 - \omega^2 - \mathrm{j}\omega\right)}{\left(1 - \omega^2\right)^2 + \omega^2} && (2.3) \end{aligned}$$

The amplitude response and the phase response follow from eq (2.2):

Amplitude response:

$$A\left(\mathrm{j}\omega\right) = \left|G\left(\mathrm{j}\omega\right)\right| = \frac{0.1}{\sqrt{\left(1-\omega^2\right)^2+\omega^2}}$$

Phase response:

$$\varphi\left(\omega\right) = -\arctan\left(\frac{\omega}{1-\omega^2}\right)$$

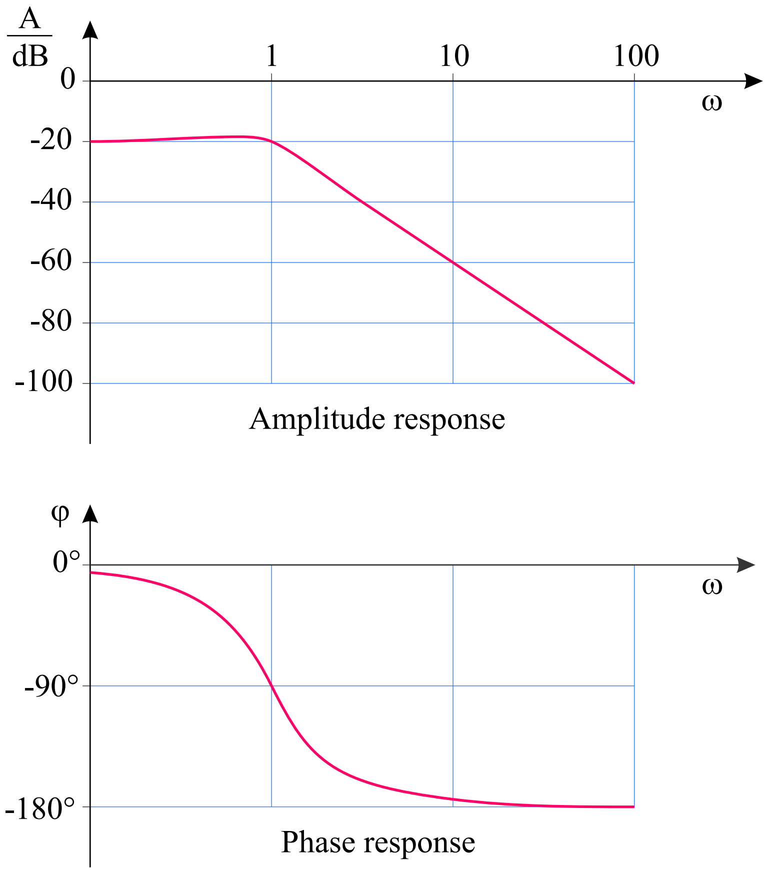


Figure 2.14: Bode diagram: Amplitude response in decibels and phase response over logarithmic frequency

The real part and the imaginary part follow from eq (2.3):

Real part:

$$\Re\left(G\left(\mathrm{j}\omega\right)\right) = \frac{0.1(1-\omega^2)}{\left(1-\omega^2\right)^2+\omega^2}$$

Imaginary part:

$$\Im\left(G\left(\mathrm{j}\omega\right)\right) = \frac{-0.1\omega}{\left(1-\omega^2\right)^2+\omega^2}$$

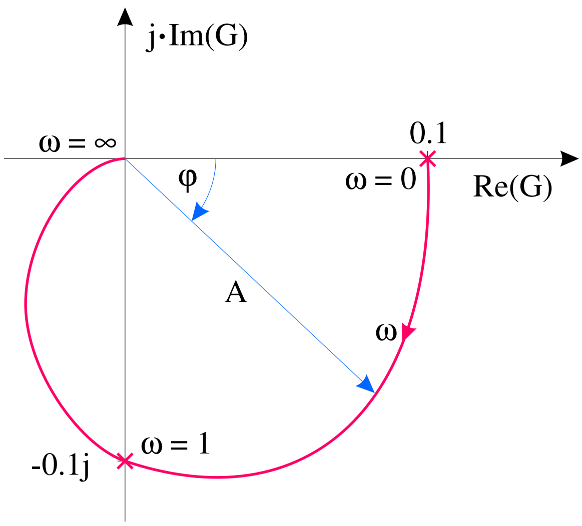


Figure 2.15: Nyquist plot: Imaginary part over real part of the frequency response with frequency as the parameter of the plot

2.7 P-element

Other names: P-controller, gain

Example: (ideal) audio amplifier

"Differential equation" in the time domain:

$$v(t) = K \cdot u(t)$$

"Differential equation" in the frequency domain:

$$V(s) = K \cdot U(s)$$

Transfer function:

$$G(s) = \frac{V(s)}{U(s)} = K$$

Poles: none

Zeroes: none

Frequency response:

$$G(j\omega) = K$$

Amplitude response:

$$A = |G(j\omega)| = K$$

Phase response:

$$\varphi = 0$$

(a) Bode diagram (b) Nyquist plot

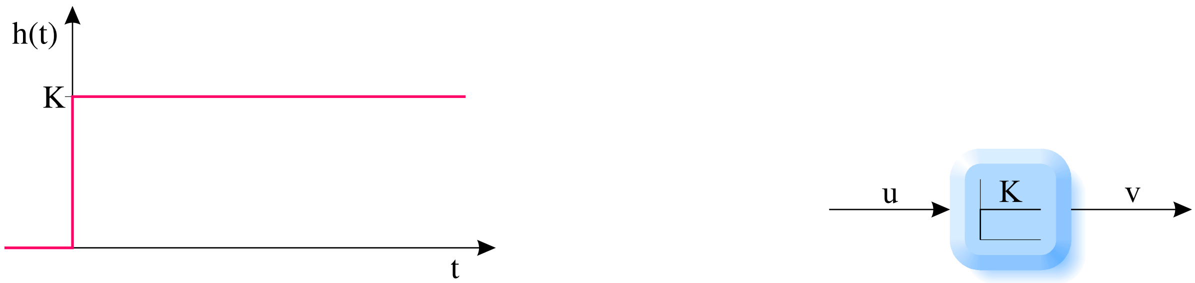
Figure 2.16: P-element

Step response in the frequency domain:

$$H(s) = G(s) \cdot \mathcal{L}\{s(t)\} = G(s) \cdot \frac{1}{s} = K \cdot \frac{1}{s}$$

Step response in the time domain (compare to table 2.1):

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{K \cdot \frac{1}{s}\right\} = K$$



(a) Step response (b) Block diagram

Figure 2.17: P-element

2.8 P-T_1

Other name: First order low pass

Example: Charging a capacitor C via a resistor R

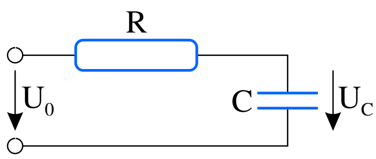


Figure 2.18: Example of a P-T_1

Differential equation in the time domain:

$$T\dot{v}(t) + v(t) = K \cdot u(t)$$

Differential equation in the frequency domain:

$$V(s) \cdot (Ts + 1) = K \cdot U(s)$$

Transfer function:

$$G(s) = \frac{V(s)}{U(s)} = \frac{K}{Ts + 1}$$

Poles:

$$Ts + 1 = 0 \quad \Rightarrow \quad s = -\frac{1}{T}$$

Zeroes: none

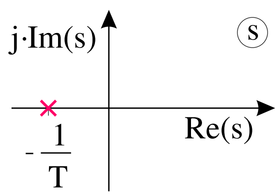


Figure 2.19: Poles (and zeros) of a (stable) P-T_1

Frequency response:

$$G(\mathrm{j}\omega) = \frac{K}{\mathrm{j}\omega T + 1}$$

Amplitude response:

$$A(\omega) = |G(\mathrm{j}\omega)| = \frac{K}{\sqrt{(\omega T)^2 + 1}}$$

Phase response:

$$\varphi(\omega) = -\arctan \omega T$$

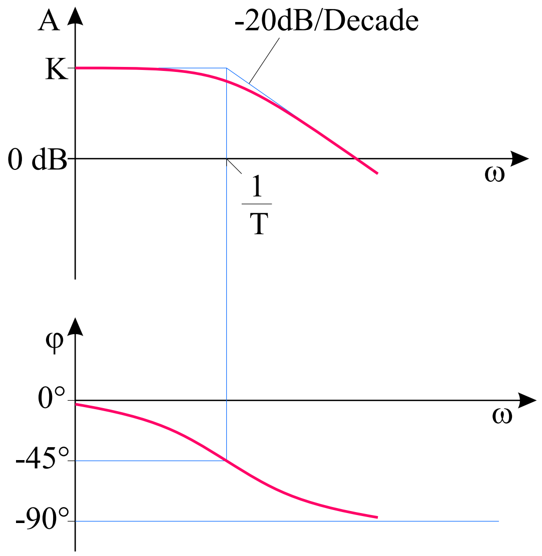


Figure 2.20: Bode diagram of a P-T_1 with corner frequency $\omega_e = {}^1\!/_T$

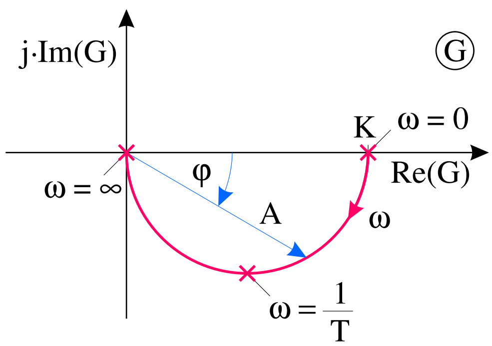


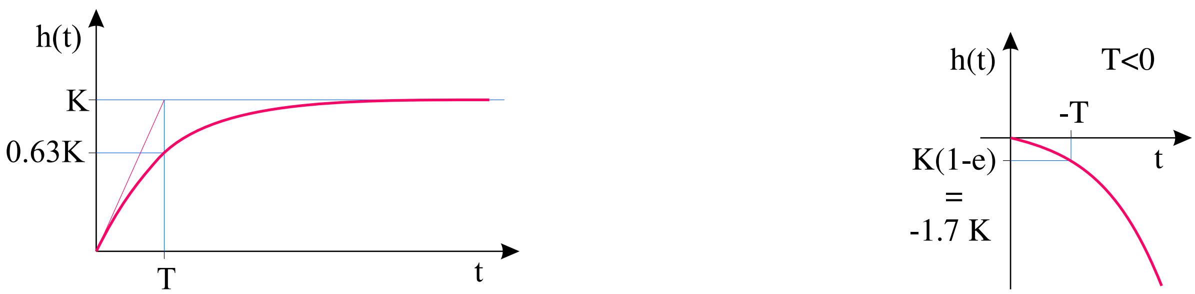
Figure 2.21: Nyquist plot of a P-T_1

Step response in the frequency domain:

$$H(s) = G(s) \cdot \frac{1}{s} = \frac{K}{Ts+1} \cdot \frac{1}{s} = \cdots \text{PFD} \cdots = \frac{K}{s} - \frac{KT}{Ts+1} = \frac{K}{s} - \frac{K}{s+\frac{1}{T}}$$

Step response in the time domain (compare table 2.1):

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = K - K\mathrm{e}^{-\frac{t}{T}} = K\left(1 - \mathrm{e}^{-\frac{t}{T}}\right)$$



(a) Stable

(b) Unstable

Figure 2.22: Step response of a P-T_1

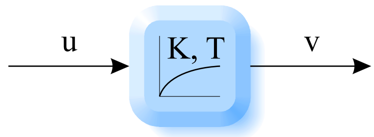


Figure 2.23: Block diagram of a P-T_1

2.9 P-T_2

Other names: Second-order low pass, Second-order oscillator

Example: Spring-mass oscillator (see section 2.1)

Differential equation in the time domain:

$$\ddot{v}(t) + 2D\omega_0\dot{v}(t) + \omega_0^2 v(t) = K\omega_0^2 u(t)$$

Differential equation in the frequency domain:

$$V(s) \cdot \left(s^2 + 2D\omega_0 s + \omega_0^2\right) = K\omega_0^2 U(s)$$

Transfer function:

$$G(s) = \frac{V(s)}{U(s)} = \frac{K\omega_0^2}{s^2 + 2D\omega_0 s + \omega_0^2}$$

Poles: see section 2.4

Zeroes: none

Frequency response:

$$G(\mathrm{j}\omega) = \frac{K\omega_0^2}{(\omega_0^2 - \omega^2) + 2D\omega_0\omega\mathrm{j}}$$

Amplitude response:

$$A(\omega) = |G(\mathrm{j}\omega)| = \frac{K\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2D\omega_0\omega)^2}}$$

Phase response:

$$\varphi(\omega) = -\arctan\left(\frac{2\,D\omega_0\omega}{\omega_0^2 - \omega^2}\right)$$

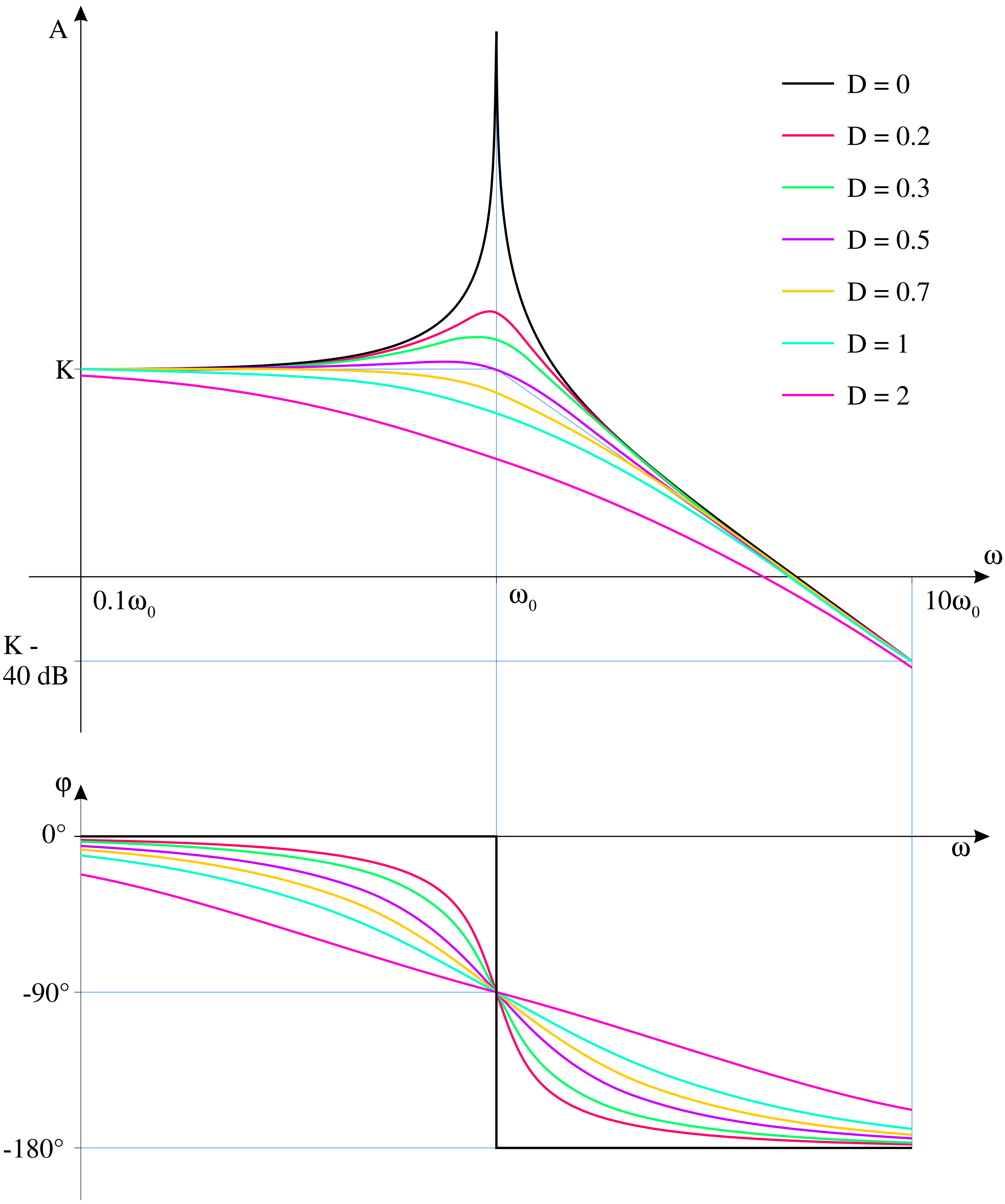


Figure 2.24: Bode diagram of a P-T_2 with different dampings

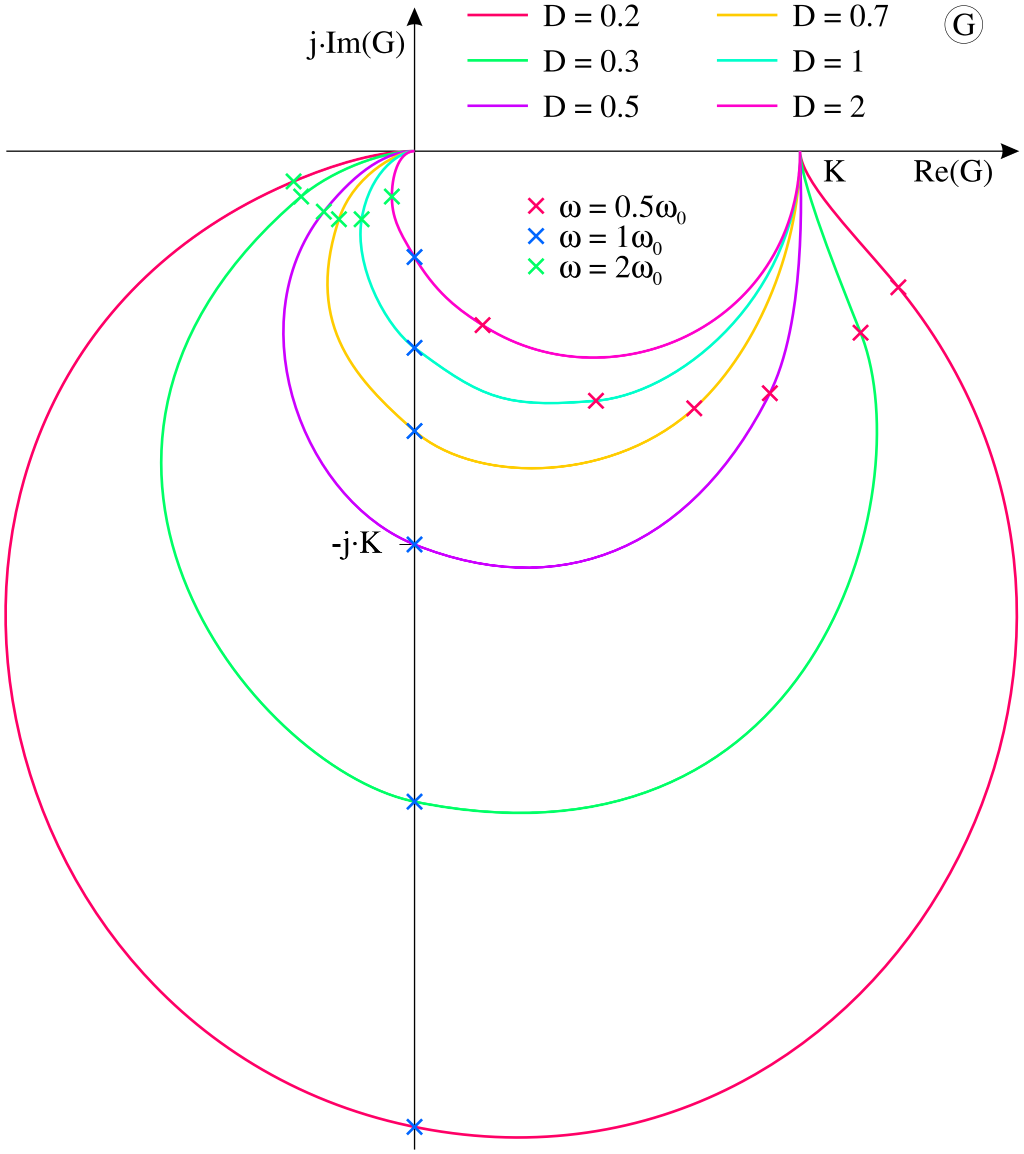


Figure 2.25: Nyquist plot of a P-T_2 with different dampings

Step response (compare eq (2.1)):

$$h(t) = K\left(1 - \mathrm{e}^{-D\omega_0 t}\left(\cos\left(\sqrt{1-D^2}\omega_0 t\right) + \frac{D}{\sqrt{1-D^2}}\sin\left(\sqrt{1-D^2}\omega_0 t\right)\right)\right)$$

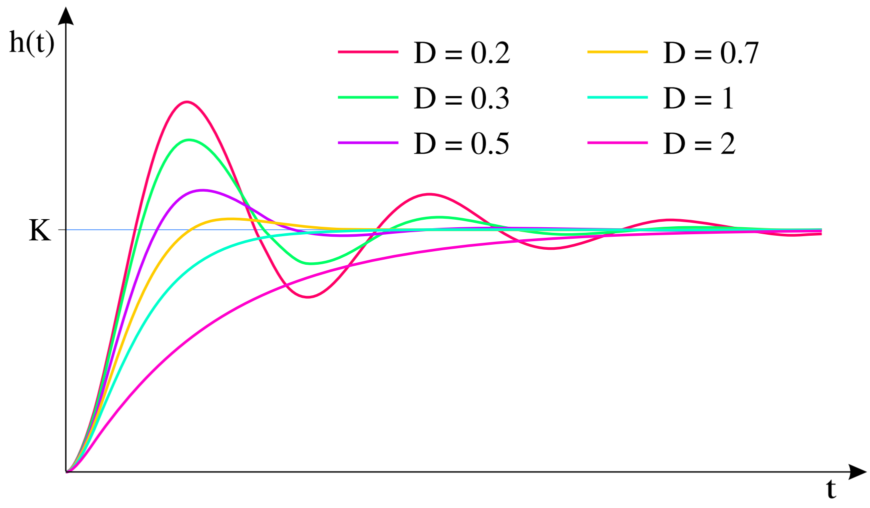


Figure 2.26: Step responses of a P-T_2 with different dampings

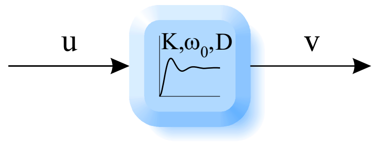


Figure 2.27: Block diagram of a P-T_2

2.10 I-element

Other names: integrator, energy storage

Example: water level in a container

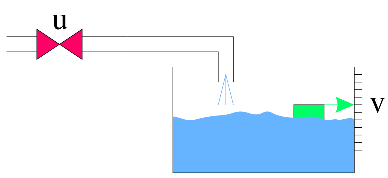


Figure 2.28: Example of an I-element

Integral equation in the time domain:

$$v(t) = \frac{1}{T_I} \int_0^t u(\tau)\,\mathrm{d}\tau + v(t = 0)$$

Differential equation in the time domain:

$$\dot{v}(t)\,T_I = u(t)$$

Differential equation in the frequency domain:

$$V(s)\, s \cdot T_I = U(s)$$

Transfer function:

$$G(s) = \frac{V(s)}{U(s)} = \frac{1}{T_I \cdot s} \left(= \frac{K_I}{s} \quad \text{with} \quad K_I = \frac{1}{T_I} \right)$$

Poles:

$$T_I s = 0 \quad \Rightarrow \quad s_1 = 0$$

Zeroes: none

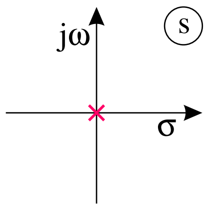


Figure 2.29: Poles (and zeros) of an I-element

Frequency response:

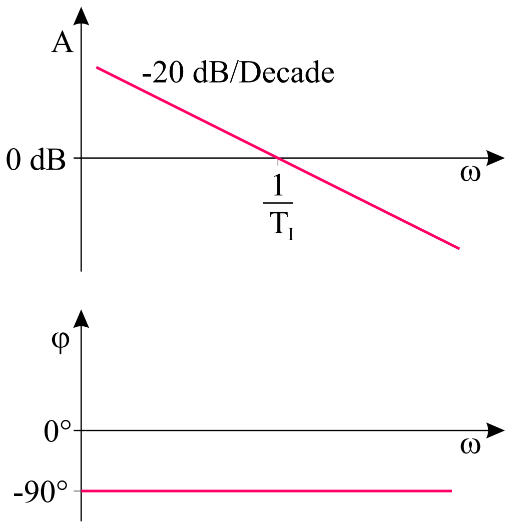
$$G(\mathrm{j}\omega) = \frac{1}{T_I \mathrm{j}\omega} = -\mathrm{j}\frac{1}{\omega T_I} = \frac{1}{\omega T_I} \mathrm{e}^{-\mathrm{j}\frac{\pi}{2}}$$

Amplitude response:

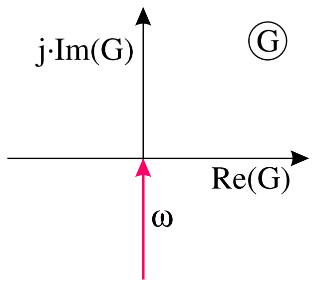
$$A(\omega) = |G(\mathrm{j}\omega)| = \frac{1}{\omega T_I}$$

Phase response:

$$\varphi(\omega) = \arctan \frac{-\frac{1}{\omega T_I}}{0} = -\arctan \infty = -\frac{\pi}{2}$$



(a) Bode diagram



(b) Nyquist plot

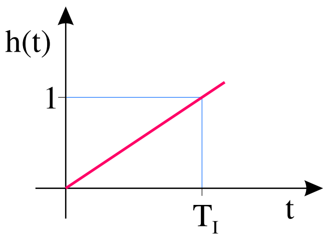
Figure 2.30: I-element

Step response in the frequency domain:

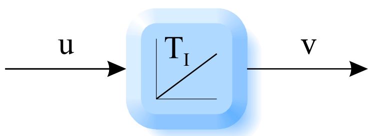
$$H(s) = G(s)\,\frac{1}{s} = \frac{1}{T_I s}\,\frac{1}{s} = \frac{1}{T_I s^2}$$

Step response in the time domain:

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{T_I s^2}\right\} = \frac{t}{T_I} = \frac{1}{T_I}t$$



(a) Step response



(b) Block diagram

Figure 2.31: I-element

2.11 D-element

Other names: differentiator

Example: the angle of rotation of the speedometer needle is the derivative of the wheel angle:

$$\varphi_{speedometer} = \dot{\varphi}_{wheel}$$

Figure 2.32: Example of a D-element

Differential equation in the time domain:

$$v(t) = T_D \dot{u}(t)$$

Differential equation in the frequency domain:

$$V(s) = T_D s U(s)$$

Transfer function:

$$G(s) = \frac{V(s)}{U(s)} = T_D s$$

Poles: none

Zeroes:

$$T_D s = 0 \quad \Rightarrow \quad s_1 = 0$$

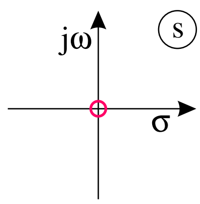


Figure 2.33: (Poles and) zeroes of a D-element

Frequency response:

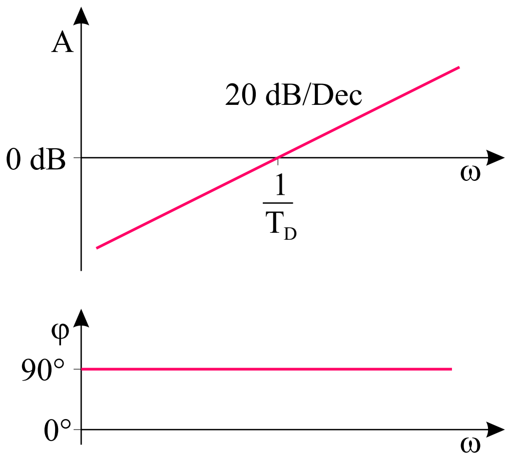
$$G(\mathrm{j}\omega) = \mathrm{j}\omega T_D = \omega T_D \mathrm{e}^{\mathrm{j}\frac{\pi}{2}}$$

Amplitude response:

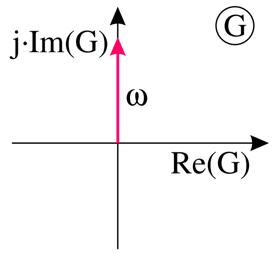
$$A(\omega) = |G(\mathrm{j}\omega)| = \omega T_D$$

Phase response:

$$\varphi(\omega) = \arctan \frac{\omega T_D}{0} = \arctan \infty = \frac{\pi}{2}$$



(a) Bode diagram



(b) Nyquist plot

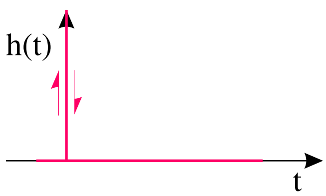
Figure 2.34: D-element

Step response in the frequency domain:

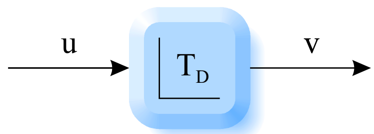
$$H(s) = G(s)\frac{1}{s} = T_D s \frac{1}{s} = T_D$$

Step response in the frequency domain (compare table 2.1):

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\{T_D\} = T_D \delta(t) \quad \text{(Dirac impulse)}$$



(a) Step response



(b) Block diagram

Figure 2.35: D-element

2.12 PID-element

Other name: PID-controller

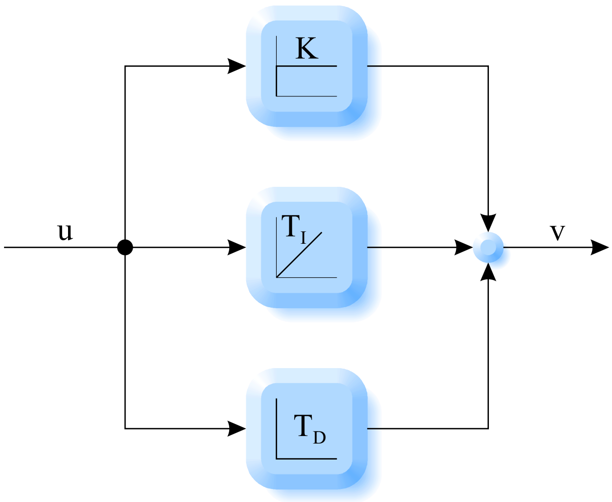


Figure 2.36: Parallel circuit of P-element, I-element and D-element

Transfer function:

$$G(s) = K + \frac{1}{T_I s} + T_D s = \frac{K T_I s + 1 + T_D T_I s^2}{T_I s}$$

with:

$$T_I = \frac{T_N}{K} \quad \text{and} \quad T_D = T_V K$$

$$G(s) = \frac{K \frac{T_N}{K} s + 1 + T_V K \frac{T_N}{K} s^2}{\frac{T_N}{K} s} = K \frac{T_N s + 1 + T_V T_N s^2}{T_N s} = K \left(1 + \frac{1}{T_N s} + T_V s\right)$$

with:

$$T_N\text{: reset time} \quad \text{and} \quad T_V\text{: rate time}$$

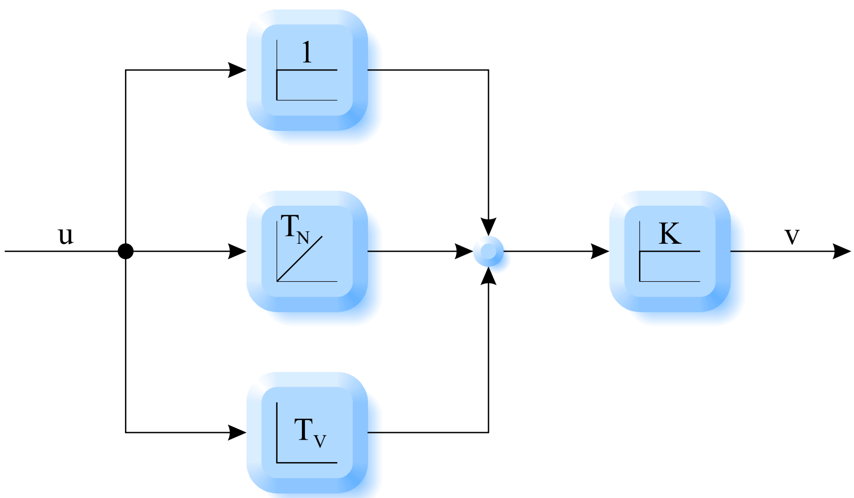


Figure 2.37: Hardware-oriented representation of a PID-element

Poles:

$$s_1 = 0$$

Zeroes:

$$T_V T_N s^2 + T_N s + 1 = 0 \quad \Rightarrow \quad \cdots \quad \Rightarrow \quad s_{1,2} = \omega_{E_{1,2}}$$

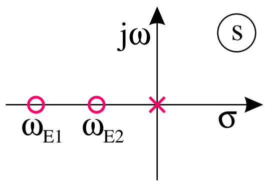


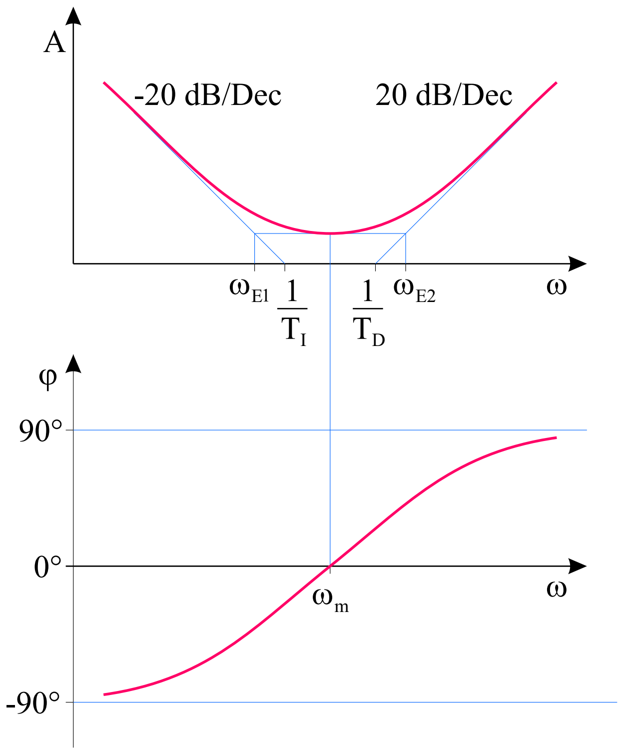
Figure 2.38: Poles and zeros of a PID-element

Frequency response:

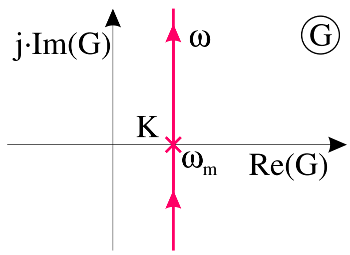
$$G(\mathrm{j}\omega) = K + \frac{1}{\mathrm{j}\omega T_I} + \mathrm{j}\omega T_D = K + \left(\frac{-1}{\omega T_I} + \omega T_D\right)\mathrm{j}$$

Intersection with the 0 degree axis:

$$\varphi = 0 \quad \Rightarrow \quad Im = 0 \quad \Rightarrow \quad \frac{-1}{\omega T_I} + \omega T_D = 0 \quad \Rightarrow \quad \frac{1}{\omega T_I} = \omega T_D \quad \Rightarrow$$
$$\omega_m = \sqrt{\frac{1}{T_I \cdot T_D}} = \sqrt{\frac{1}{\frac{T_N}{K} \cdot T_V K}} = \sqrt{\frac{1}{T_N \cdot T_V}}$$

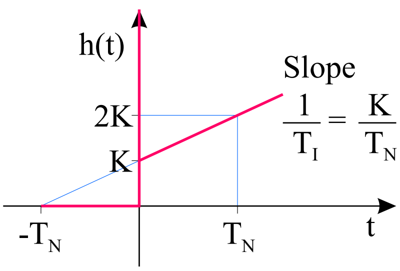


(a) Bode diagram

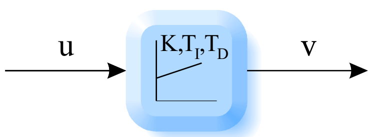


(b) Nyquist plot

Figure 2.39: PID-element



(a) Step response



(b) Block diagram

Figure 2.40: PID-element

2.13 Dead time

Other name: time delay

Example: conveyor belt

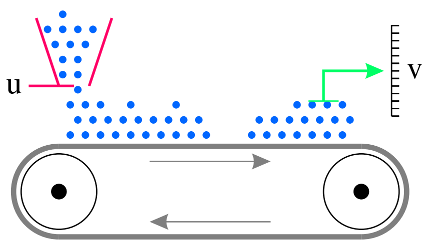


Figure 2.41: Conveyor belt as an example of a dead time

Transfer function:

$$G\left(s\right)=\mathrm{e}^{-sT_T}$$

Poles: none

Zeroes: none

Frequency response:

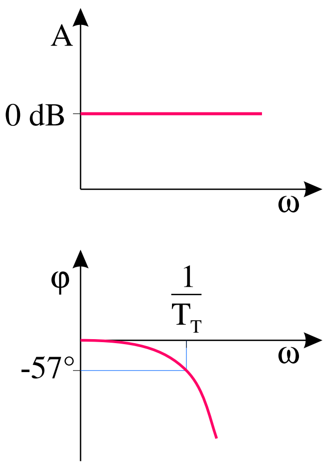
$$G\left(\mathrm{j}\omega\right)=\mathrm{e}^{-\mathrm{j}\omega T_T}$$

Amplitude response:

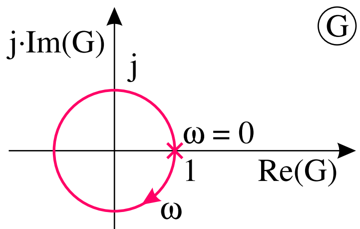
$$A\left(\omega\right)=\left|G\left(\mathrm{j}\omega\right)\right|=1$$

Phase response:

$$\varphi\left(\omega\right)=-\omega T_T$$

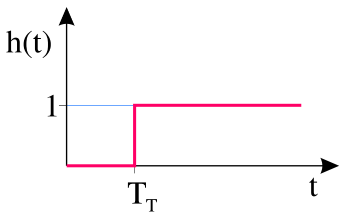


(a) Bode diagram

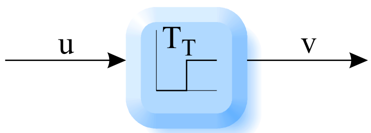


(b) Nyquist plot

Figure 2.42: Dead time



(a) Step response



(b) Block diagram

Figure 2.43: Dead time

2.14 State space representation

Differential equation of a second-order system:

$$\ddot{v} + 2D\omega_0\dot{v} + \omega_0^2 v = K\omega_0^2 u$$

Introduction of 2 state variables:

$$\begin{aligned} x_1 &= v \quad \text{(Displacement)} \\ x_2 &= \dot{v} \quad \text{(Velocity)} \end{aligned}$$

First differential equation:

$$\dot{x}_1 = x_2$$

Second differential equation:

$$\begin{aligned} \dot{x}_2 + 2D\omega_0 x_2 + \omega_0^2 x_1 &= K\omega_0^2 u \\ \dot{x}_2 &= -\omega_0^2 x_1 - 2D\omega_0 x_2 + K\omega_0^2 u \end{aligned}$$

Matrix notation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2D\omega_0 \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ K\omega_0^2 \end{bmatrix}}_{\boldsymbol{B}=\boldsymbol{b}} u$$

Vector differential equation in state space:

$$\dot{\boldsymbol{x}} = \boldsymbol{A} \cdot \boldsymbol{x} + \boldsymbol{B} \cdot \boldsymbol{u}$$

Vector output equation:

$$\boldsymbol{v} = \boldsymbol{C} \cdot \boldsymbol{x} + \boldsymbol{D} \cdot \boldsymbol{u}$$

Output equation in explicit form:

$$v = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\boldsymbol{C}=\boldsymbol{c}^T} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{\boldsymbol{D}=d} u$$

2.14.1 Block diagram of the state space representation

- ✓ "One integrator per state"
- ✓ "Model the differential equation at the input of the integrator"

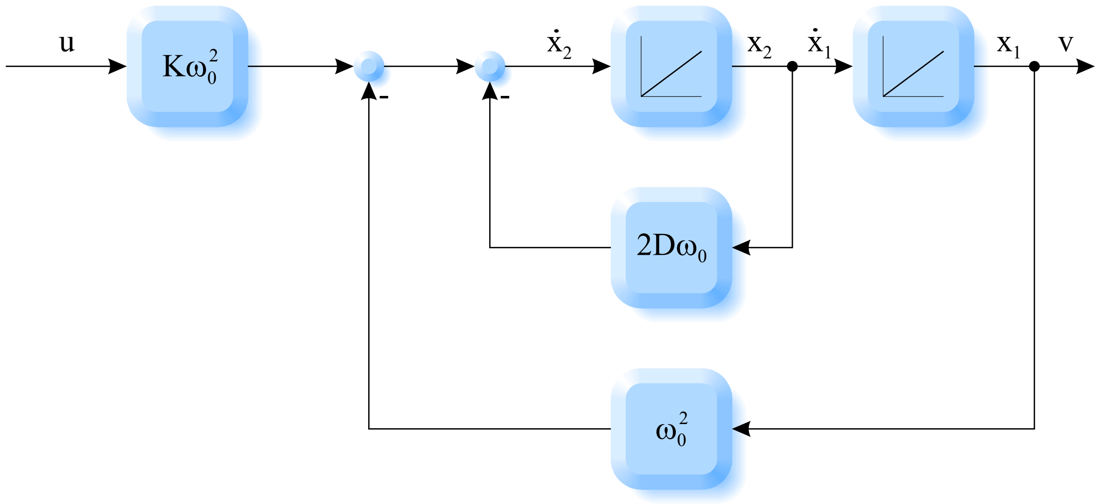


Figure 2.44: Block diagram of a second-order oscillator

Chapter 3

Controller design

3.1 Stability

3.1.1 BIBO-stability (Bounded Input Bounded Output)

"Stable when a limited input signal leads to a limited output signal."

3.1.1.1 Example: step response of an integrator

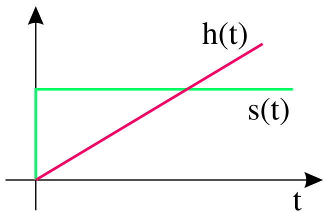


Figure 3.1: Example: step response of an integrator

Result: integrator is not stable.

3.1.2 Asymptotic stability

- stable if the impulse response asymptotically decays to zero.
- unstable if the impulse response approaches infinity
- marginally stable if the impulse response does not exceed a finite value

3.1.2.1 Example: impulse response of an integrator

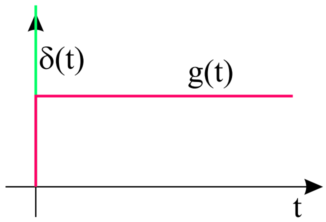


Figure 3.2: Example: impulse response of an integrator

Result: integrator is marginally stable.

3.1.2.2 Example: impulse response of a double integrator

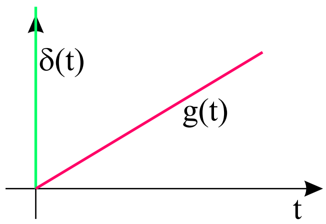


Figure 3.3: Example: impulse response of a double integrator

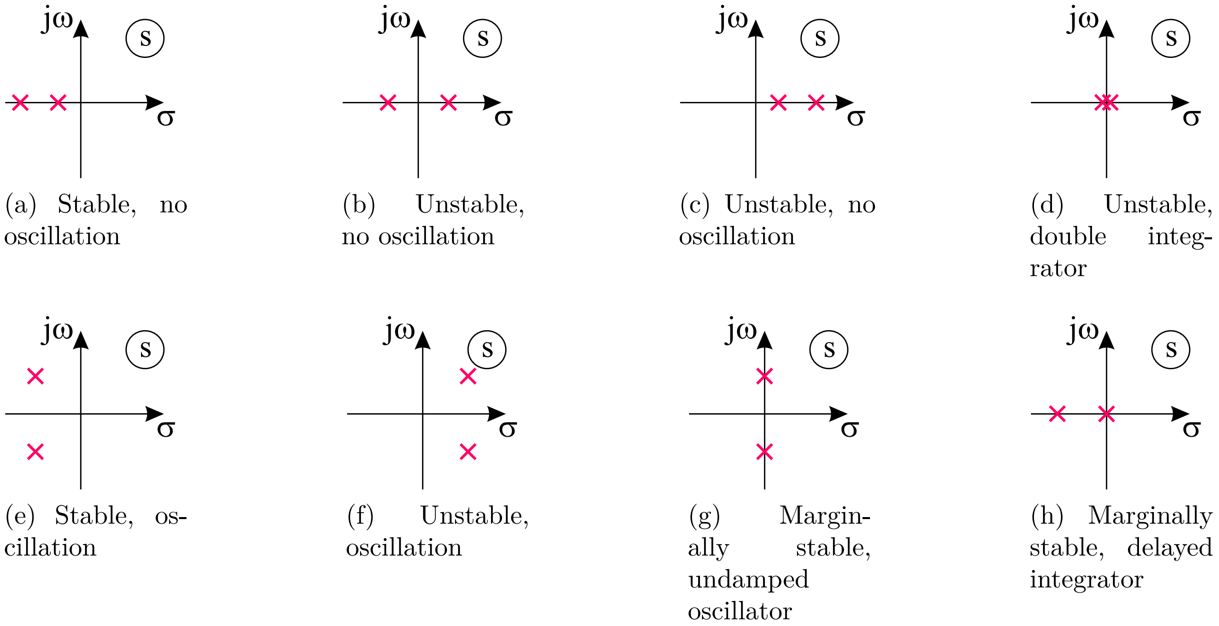
Result: double integrator is unstable.

3.1.3 Fundamental stability criterion

- stable if the transfer function only has poles in the left half plane
- unstable if at least one pole lies in the right half plane or if at least one multiple pole lies on the imaginary axis
- marginally stable if no pole lies in the right half plane, there are no multiple poles on the imaginary axis, but there is at least one simple pole on the imaginary axis

3.1.3.1 Example: general second-order system

$$G(s) = \frac{1}{s^2 + a_1 s + a_0} \quad \Rightarrow \quad s_{1,2} = \ldots$$



(a) Stable, no oscillation

(b) Unstable, no oscillation

(c) Unstable, no oscillation

(d) Unstable, double integrator

(e) Stable, oscillation

(f) Unstable, oscillation

(g) Marginally stable, undamped oscillator

(h) Marginally stable, delayed integrator

Figure 3.4: Pole distributions of stable and unstable systems

3.2 Control loop

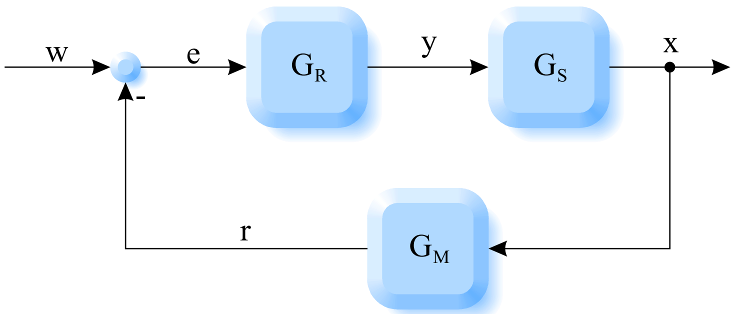


Figure 3.5: General control loop

“From the output backwards to all inputs, or to the output itself”:

$$x = G_S \cdot y = G_S \cdot G_R \cdot e = G_S \cdot G_R \cdot (w - r) = G_S \cdot G_R \cdot (w - G_M \cdot x)$$

Sort:

$$x \cdot (1 + G_S \cdot G_R \cdot G_M) = G_S \cdot G_R \cdot w$$

Overall transfer function:

$$G_g = \frac{x}{w} = \frac{G_S \cdot G_R}{1 + G_S \cdot G_R \cdot G_M} = \frac{G_V}{1 + G_0}$$

Forward transfer function:

$$G_V = G_S \cdot G_R$$

Open loop transfer function:

$$G_0 = G_S \cdot G_R \cdot G_M$$

3.2.1 Example

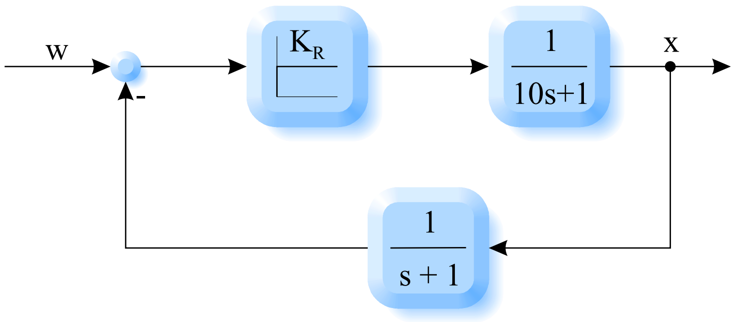


Figure 3.6: Example of a simple control loop

Overall transfer function:

$$G_g = \frac{G_V}{1+G_0} = \frac{K_R \cdot \frac{1}{10s+1}}{1 + K_R \cdot \frac{1}{10s+1} \cdot \frac{1}{s+1}} = \frac{K_R\,(s+1)}{(10s+1)\,(s+1) + K_R}$$

Poles:

$$10s^2 + 10s + s + 1 + K_R = 0$$

Normal form:

$$s^2 + \frac{11}{10}s + \frac{1+K_R}{10} = 0$$

Two (real or complex conjugate) poles:

$$s_{1,2} = -\frac{11}{20} \pm \sqrt{\left(\frac{11}{20}\right)^2 - \frac{1+K_R}{10}}$$

For the special controller gain $K_R = 1$:

$$s_{1,2} = -\frac{11}{20} \pm \sqrt{\left(\frac{11}{20}\right)^2 - \frac{2}{10}}$$

there are two real (stable) poles:

$$s_1 = -0.2 \quad \text{und} \quad s_2 = -0.87$$

For the special controller gain $K_R = 10$:

$$s_{1,2} = -\frac{11}{20} \pm \sqrt{\left(\frac{11}{20}\right)^2 - \frac{11}{10}}$$

there are two complex conjugate (stable) poles:

$$s_{1,2} = -0.55 \pm 0.89\mathrm{j}$$

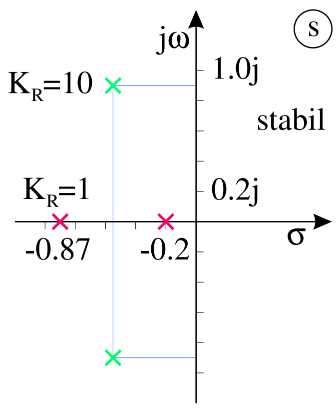


Figure 3.7: Poles of a simple control loop depending on the controller gain

3.3 Nyquist criterion

Preliminary consideration:

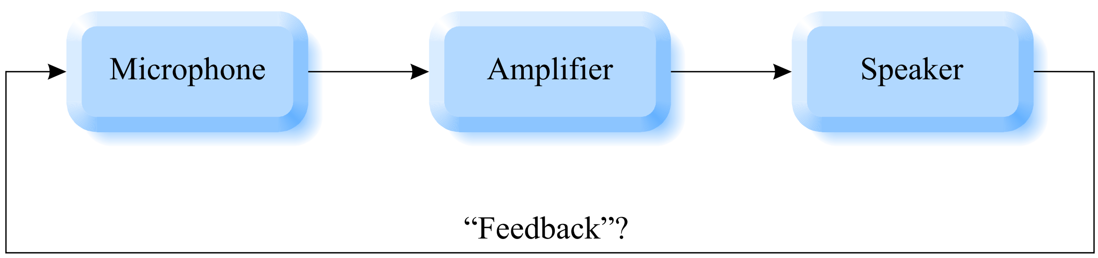


Figure 3.8: When does the acoustic loop become unstable?

Feedback only if :

1. gain large enough (> 1) and
2. positive feedback present (phase shift $= n \cdot 2\pi$)

Control loop:

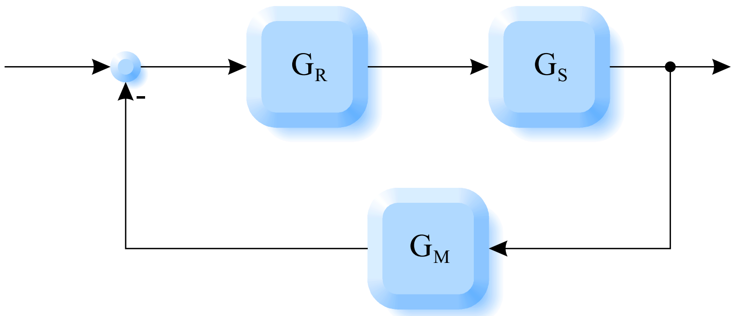


Figure 3.9: General control loop

Open loop transfer function:

$$G_0 = G_R \cdot G_S \cdot G_M$$

Critical point:

$$A = |G_0| = 1 \quad \text{und} \quad \varphi = \angle G_0 = -\pi$$

3.3.1 Example

P-controller

$$G_R = K_R$$

P-T_2 plant:

$$G_S = \frac{0.1}{s^2 + s + 1}$$

P-T_1 sensor:

$$G_M = \frac{1}{0.1s + 1}$$

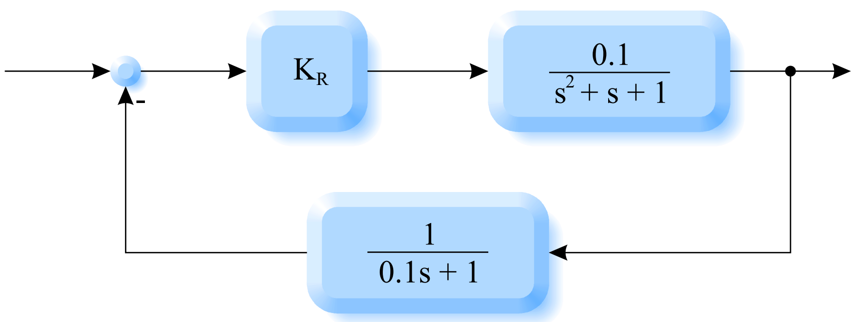


Figure 3.10: Block diagram of a control loop with P-controller, P-T_2-plant and P-T_1-sensor

Open loop transfer function:

$$G_0 = G_R \cdot G_S \cdot G_M = K_R \cdot \frac{0.1}{s^2+s+1} \cdot \frac{1}{0.1s+1}$$

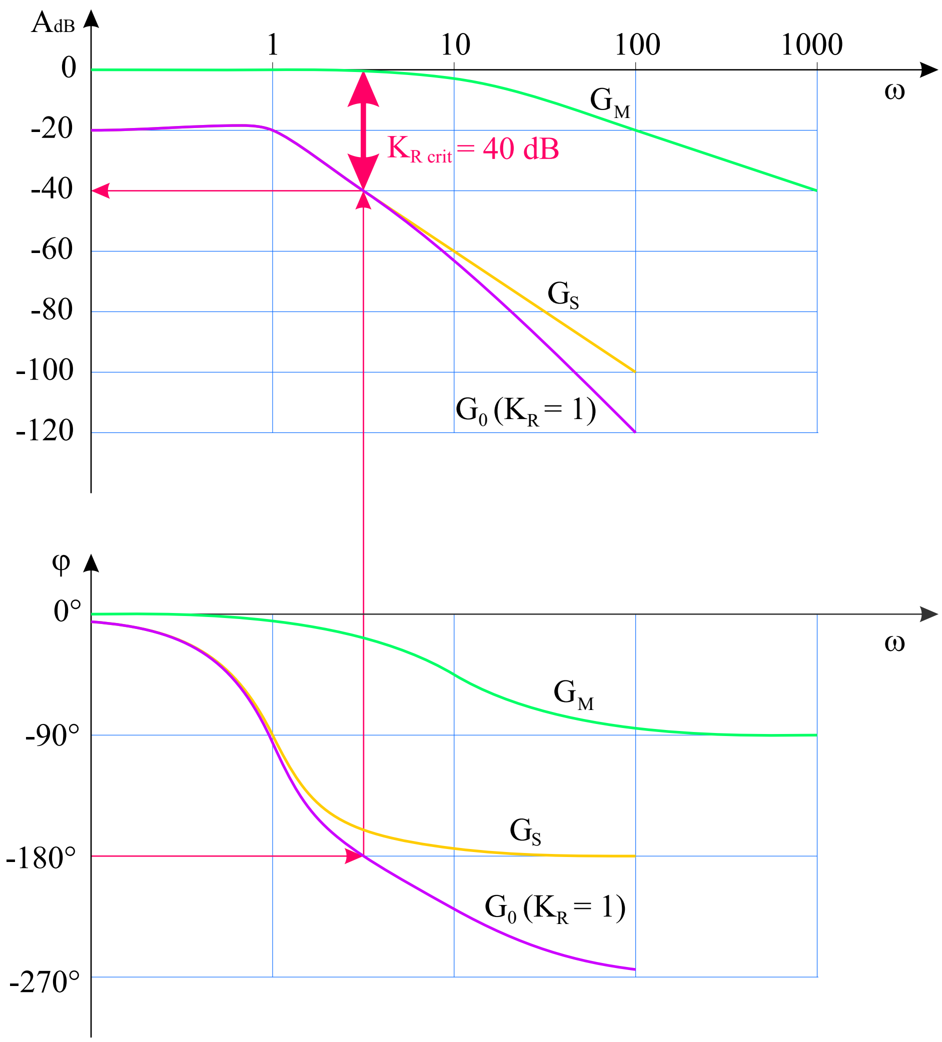


Figure 3.11: Bode diagram of the open loop transfer function (G_0)

Shift up to 0 dB (amplitude reserve):

$$K_{R_{crit}} \approx 40\,\mathrm{dB} = 100 \quad \text{(closed loop is marginally stable)}$$

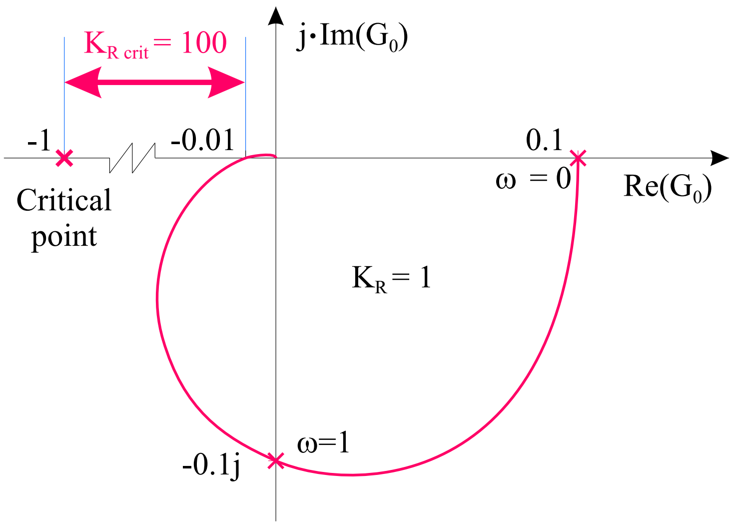


Figure 3.12: Nyquist plot of the open loop transfer function (G_0)

Simplified Nyquist criterion on the Nyquist curve of the *open* loop: Critical point (-1) must be on the left, then the *closed* circle G_g is stable! (Requirement: stable open loop plus a maximum of two integrators)

3.3.2 Two ways to examine the stability of the control loop

1. calculate the *closed* circle transfer function and apply the **fundamental stability criterion** to the closed circle (poles of the closed circle in the left half plane)
2. calculate the *open* loop transfer function ($G_0 = G_R G_S G_M$) and apply the **Nyquist criterion**

3.4 Controller design

Three (sometimes contradictory) demands:

1. stability
2. speed
3. small control error

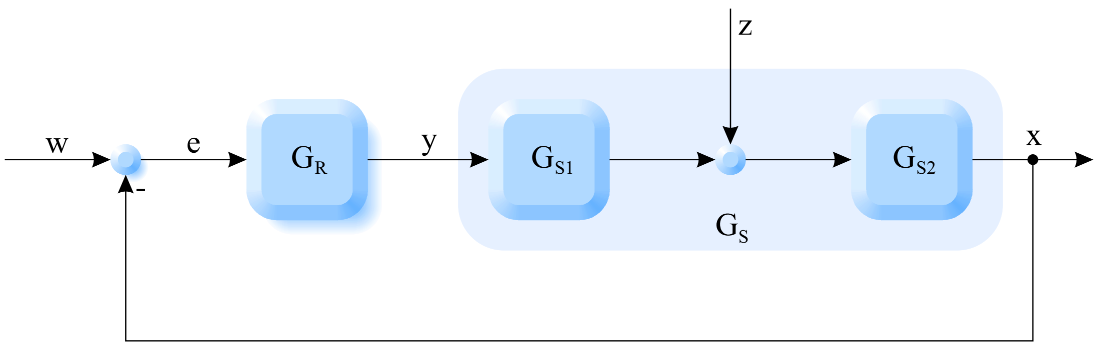


Figure 3.13: General control loop with disturbance and segmented plant

Open loop transfer functions:

$$G_0 = G_R \cdot G_{S_1} \cdot G_{S_2}$$

Reference transfer function:

$$G_g = \frac{x}{w} = \frac{G_0}{1 + G_0} \quad \text{(preferably} \to 1\text{)}$$

Disturbance transfer function:

$$G_z = \frac{x}{z} = \frac{G_{S_2}}{1 + G_0} \quad \text{(preferably} \to 0\text{)}$$

Reference control error:

$$G_{ew} = \frac{e}{w} = \frac{1}{1 + G_0} \quad \text{(preferably} \to 0\text{)}$$

Disturbance control error:

$$G_{ez} = \frac{e}{z} = -\frac{G_{S_2}}{1 + G_0} \quad \text{(preferably} \to 0\text{)}$$

3.4.1 Example: P-T$_3$

$$G_{S_1} = \frac{1}{s + 1}$$
$$G_{S_2} = \frac{2}{s^2 + s + 1}$$

P-controller:

$$G_R = K_R$$

Open loop transfer function:

$$G_0 = \frac{2K_R}{(s + 1)\,(s^2 + s + 1)}$$

Closed loop transfer function:

$$G_g = \frac{G_0}{1 + G_0} = \frac{\frac{2K_R}{(s+1)(s^2+s+1)}}{1 + \frac{2K_R}{(s+1)(s^2+s+1)}} = \frac{2K_R}{(s + 1)\,(s^2 + s + 1) + 2K_R}$$

Good control accuracy:

$$K_R \to \infty \quad \Rightarrow \quad G_g \to 1$$

But: if K_R is too large $\Rightarrow$ control loop might become unstable.
Steady-state control error transfer function:

$$G_{ew} = \frac{1}{1+G_0} = \frac{1}{1+\frac{2K_R}{(s+1)(s^2+s+1)}} = \frac{(s+1)(s^2+s+1)}{(s+1)(s^2+s+1)+2K_R}$$

Small control error:

$$K_R \to \infty \quad \Rightarrow \quad G_{ew} \to 0$$

But: see above
Limit theorem of the Laplace transform:

$$\lim_{t\to\infty} h(t) = \lim_{s\to 0} G(s)$$

Reference step:

$$w = s(t)$$

Steady-state control error:

$$\lim_{t\to\infty} e = e_\infty = \lim_{s\to 0} G_{ew}$$

$$e_\infty = \lim_{s\to 0} \frac{(s+1)(s^2+s+1)}{(s+1)(s^2+s+1)+2K_R} = \frac{1}{1+2K_R}$$

Increasing the controller gain reduces the steady-state control deviation:

$$K_R \uparrow \quad \Rightarrow \quad e_\infty \downarrow$$

When using an I-controller:

$$G_R = \frac{1}{T_I \cdot s}$$

Open loop transfer function:

$$G_0 = \frac{2}{T_I s(s+1)(s^2+s+1)}$$

Steady-state control error transfer function:

$$G_{ew} = \frac{1}{1+G_0} = \frac{1}{1+\frac{2}{T_I s(s+1)(s^2+s+1)}} = \frac{T_I s(s+1)(s^2+s+1)}{T_I s(s+1)(s^2+s+1)+2}$$

Steady-state control error:

$$e_\infty = \lim_{s\to 0} G_{ew} = \frac{T_I 0(0+1)(0^2+0+1)}{T_I 0(0+1)(0^2+0+1)+2} = \frac{0}{2} = 0$$

No steady-state control error with I-controller (disadvantage: slower, destabilization)

3.5 Quality criteria

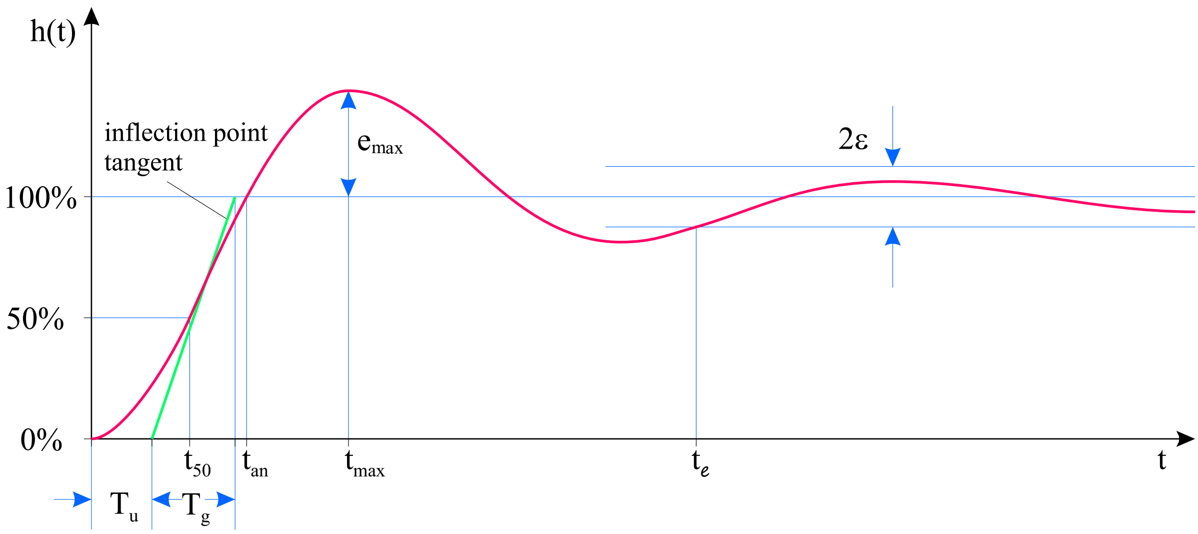


Figure 3.14: Reference step response

e_{max} maximum overshoot

t_{max} e_{max} occurs.

T_u delay time (inflection point tangent $\cap$ 0 %)

T_g compensation time (inflection point tangent $\cap$ 0 % $\cap$ 100 %)

t_{an} response time (curve $\cap$ 100 %)

t_ε the magnitude of the control error stays less than ε (z. B. $t_{3\,\%}$)

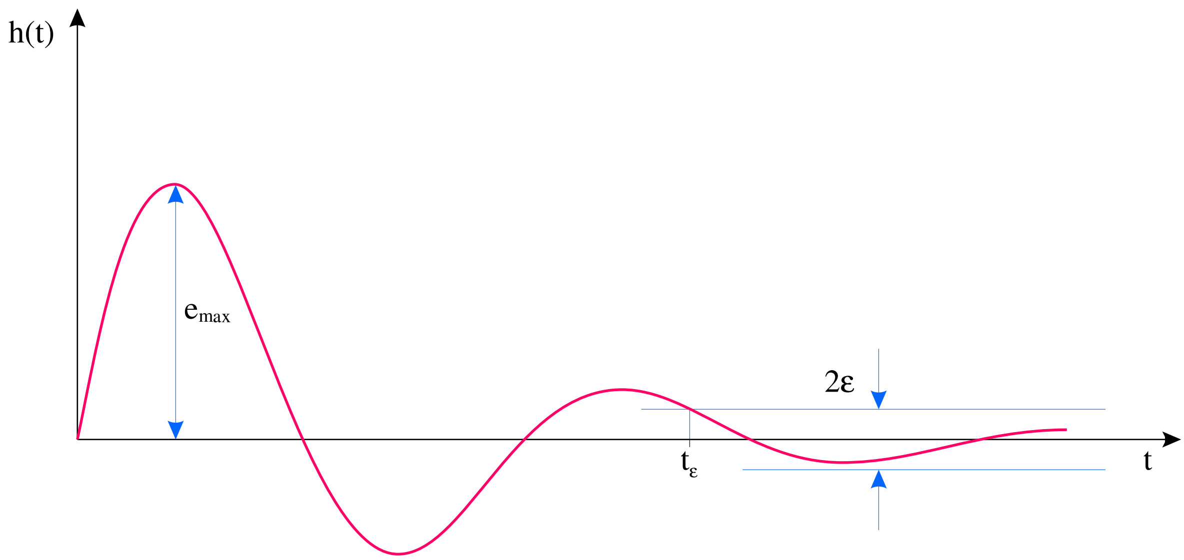


Figure 3.15: Disturbance step response

Cost function:

$$K = K_1 \cdot t_{an} + K_2 \cdot t_\varepsilon + K_3 \cdot e_{max} + K_4 \cdots = \text{minimum} \downarrow$$

$\Rightarrow \quad K_i$ arbitrarily chosen $\quad \Rightarrow \quad$ compromise

3.5.1 Integral criteria

$$I = \min \downarrow$$

Integral Error (IE):

$$I = \int_0^\infty e(t) \cdot \mathrm{d}t \quad (e > 0)$$

Integral Absolute Error (IAE):

$$I = \int_0^\infty |e(t)| \cdot \mathrm{d}t \quad \text{(inefficient)}$$

Integral Square Error (ISE):

$$I = \int_0^\infty e^2(t) \cdot \mathrm{d}t \quad \text{(analytical calculation)}$$

Integral Time Square Error (ITSE):

$$I = \int_0^\infty e^2(t) \cdot t \cdot \mathrm{d}t \quad \text{(duration of the control error)}$$

Control effort:

$$I = \int_0^\infty \left(e^2(t) + \alpha \cdot y^2(t)\right) \cdot \mathrm{d}t \quad (\alpha\text{: subjective weighting factor})$$

3.6 Controller optimisation with simulation

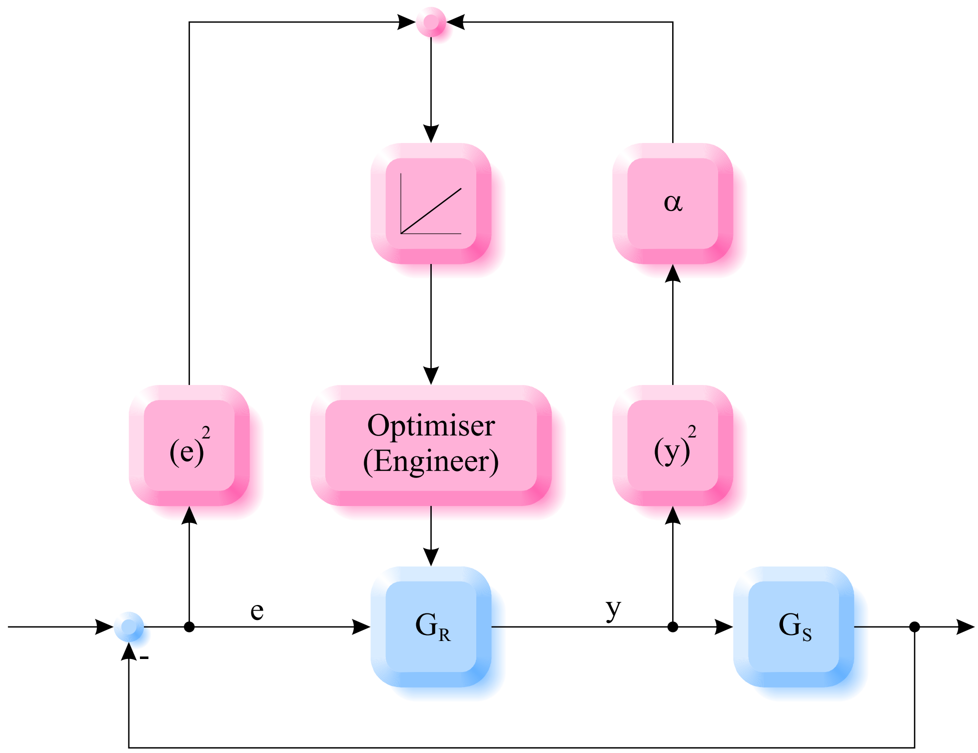


Figure 3.16: Controller optimisation

3.7 Control loop tuning rules

PID-controller (compare section 2.12)

P basic control

I steady-state accuracy (slow)

D speed (I compensation)

Two proven methods:

3.7.1 Ziegler and Nichols (margin of stability)

Turn up the P-controller until the control loop oscillates stationary:

$$\rightarrow \quad K_{R_{krit}}$$

Measure the period of the oscillation:

$$\rightarrow \quad T_{krit}$$

Controller	Controller gain K_R	Reset time T_N	Rate time T_V
P	$0.5\,K_{R_{krit}}$	-	-
PI	$0.45\,K_{R_{krit}}$	$0.85\,T_{krit}$	-
PID	$0.6\,K_{R_{krit}}$	$0.5\,T_{krit}$	$0.12\,T_{krit}$

Table 3.1: Controller gain, reset time, and rate time depending on the parameters of the oscillation test.

3.7.2 Chien, Hrones, and Reswick (step response)

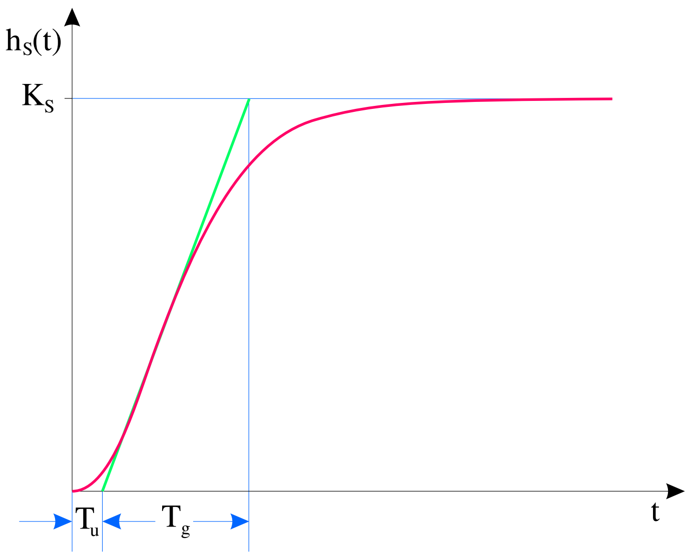


Figure 3.17: Step response of the plant

- inflection point tangent → delay time T_u, compensation time T_g
- applicable if $T_g/T_u > 3$

Definition of an auxiliary variable:

$$K_H = \frac{T_g}{K_S T_u}$$

Controller	Optimized for	Overshoot	K_R	T_N	T_V
P	disturbance	0 %	$0.3\,K_H$	-	-
		20 %	$0.7\,K_H$	-	-
	reference	0 %	$0.3\,K_H$	-	-
		20 %	$0.7\,K_H$	-	-
PI	disturbance	0 %	$0.6\,K_H$	$4.0\,T_u$	-
		20 %	$0.7\,K_H$	$2.3\,T_u$	-
	reference	0 %	$0.35\,K_H$	$1.2\,T_g$	-
		20 %	$0.6\,K_H$	$1.0\,T_g$	-
PID	disturbance	0 %	$0.95\,K_H$	$2.4\,T_u$	$0.42\,T_u$
		20 %	$1.2\,K_H$	$2.0\,T_u$	$0.42\,T_u$
	reference	0 %	$0.6\,K_H$	$1.0\,T_g$	$0.5\,T_u$
		20 %	$0.95\,K_H$	$1.35\,T_g$	$0.47\,T_u$

Table 3.2: Controller gain, reset time, and rate time depending on steady-state amplitude, delay time, and compensation time.

3.8 Disturbance feedforward

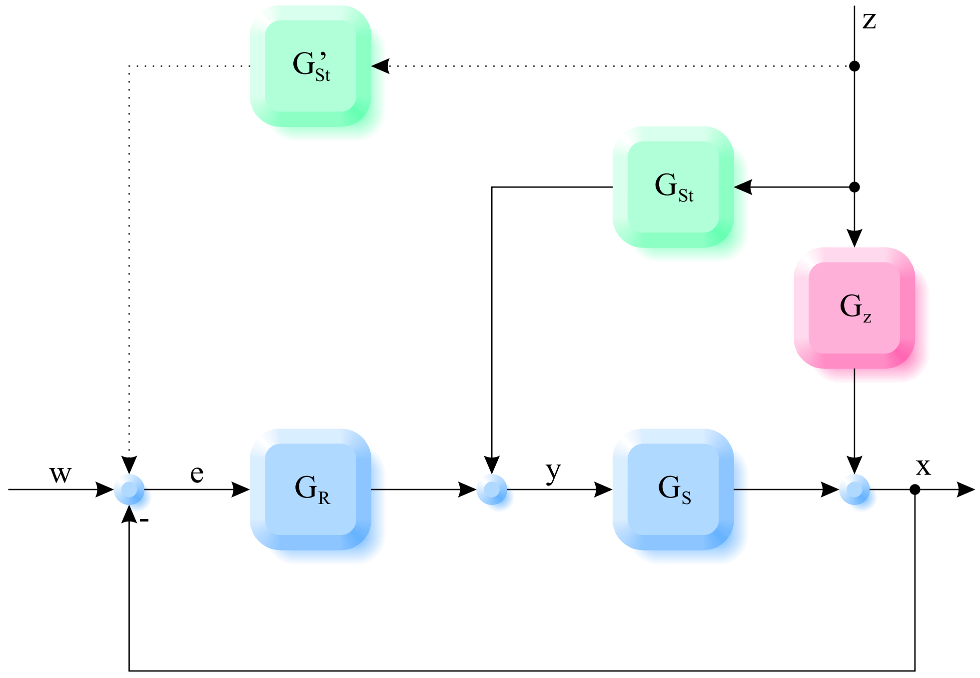


Figure 3.18: Disturbance feedforward

Concept: compensation for the disturbance variable z if it can be measured. If possible, the disturbance variable should have no influence on the output variable x:

Transfer function from z to x:

$$G_{xz} = \frac{G_z + G_{St} G_S}{1 + G_R G_S} \overset{!}{=} 0$$

Numerator of G_{xz} must disappear:

$$G_z + G_{St} G_S = 0$$

Condition for disturbance feedforward transfer function:

$$G_{St} = -\frac{G_z}{G_S}$$

Problem: G_{St} cannot always be implemented exactly

$\Rightarrow$ at least stationary disturbance compensation ($s = 0$)

Alternative: signal injection before the controller:

New transfer function from z to x:

$$G_{xz} = \frac{G_z + G'_{St} G_R G_S}{1 + G_R G_S} \overset{!}{=} 0$$

Numerator must be zero:

$$G_z + G'_{St} G_R G_S = 0$$

Compensation condition:

$$G'_{St} = -\frac{G_z}{G_R G_S}$$

Advantage: no actuator energy necessary

Disadvantage: signal has to go through the controller

3.9 Open loop control (feedforward)

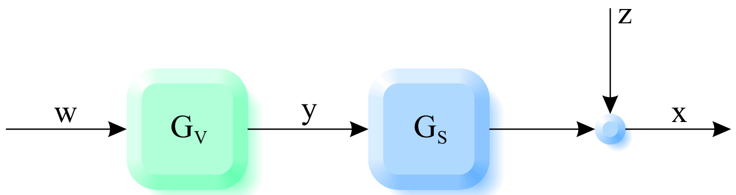


Figure 3.19: Pure open loop control

Requirement:

$$x = w$$

Ideal:

$$G_V = \frac{1}{G_S}$$

e. g.:

$$G_S = \frac{2}{3s+1} \quad \Rightarrow \quad G_V = \frac{3s+1}{2}$$

Problems:

1. G_S is not exactly known.
2. G_S is not exactly invertible (dead time, pure integrator, . . .).
3. disturbances are not detected.

Steady-state open loop control $G_{V_{st}}$ meets the requirement $x = w$ at least after the transient response ($t \to \infty \quad \Rightarrow \quad s = 0$):

Example above:

$$G_{V_{st}} = \lim_{s \to 0} \frac{3s+1}{2} = \frac{3 \cdot 0 + 1}{2} = 0.5$$

3.9.1 Combination with closed loop control

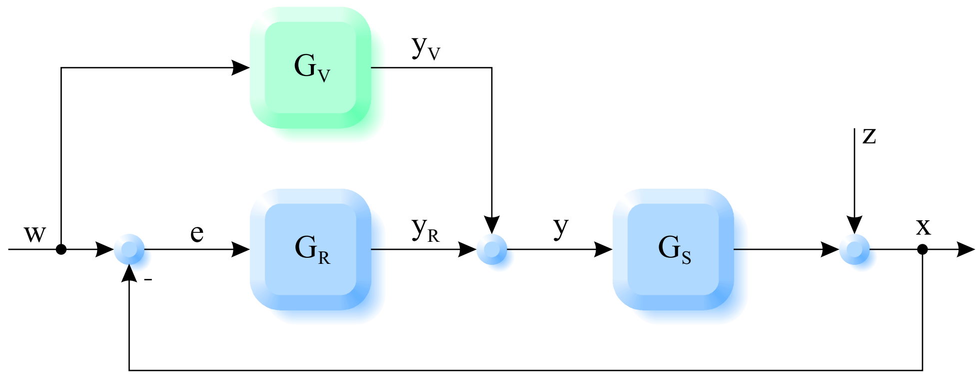


Figure 3.20: Combination of open loop and closed loop control

Method:

1. design the open loop controller G_V for best possible inversion.
2. design the closed loop controller G_R (just has to compensate the inadequacies of the open loop controller).

3.10 Digital (discrete-time) control

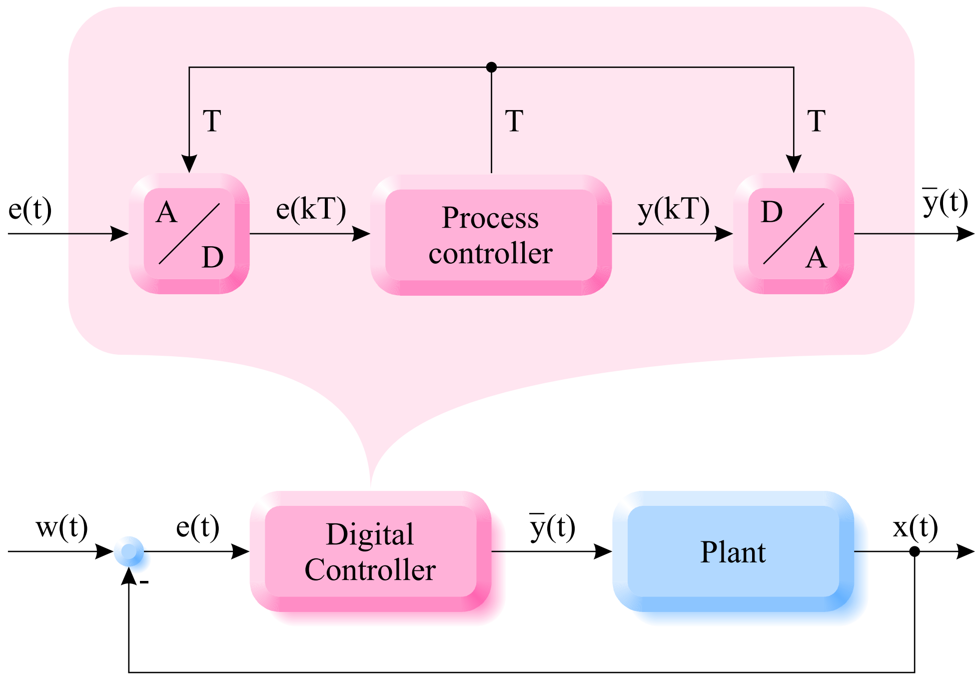


Figure 3.21: Digital control (T: sampling time)

(a) Sampled input variable plot

(b) Output variable is hold

Figure 3.22: Sampling with digital control

3.10.1 z-transformation

The transfer function in the s-domain is given:

$$G(s) = \frac{V(s)}{U(s)}$$

We are looking for the transfer function in the z-domain:

$$G(z) = \frac{v(kT)}{u(kT)}$$

and derived from this, the difference equation:

$$v_{k+1} = f\left(v_k,\, v_{k-1}, \ldots,\, u_{k+1},\, u_k,\, u_{k-1}, \ldots\right)$$

3.10.2 Approximations

Rectangle rule:

$$s \approx \frac{z-1}{T \cdot z}$$

Tustin's rule:

$$s \approx \frac{2}{T} \cdot \frac{z-1}{z+1}$$

3.10.3 Example: digital low pass

Transfer function in the s-domain:

$$G(s) = \frac{2}{5s+1}$$

Sampling time:

$$T = 0.1$$

Using Tustin's rule:

$$s \approx 20 \cdot \frac{z-1}{z+1}$$

Transfer function in the z-domain:

$$G(z) = \frac{2}{5 \cdot 20 \cdot \frac{z-1}{z+1} + 1} = \frac{2(z+1)}{100(z-1) + z + 1} = \frac{2z+2}{101z-99}$$

Quotient of output and input variable:

$$G(z) = \frac{v_k}{u_k} = \frac{2z+2}{101z-99}$$

Multiply crosswise:

$$v_k (101z - 99) = u_k (2z + 2)$$

"Multiplication by z means shifting in the positive time direction":

$$v_k \cdot z \mathrel{\hat{=}} v_{k+1}$$

Difference equation:

$$101 v_{k+1} - 99 v_k = 2u_{k+1} + 2u_k$$

Solve for "new output variable":

$$v_{k+1} = \frac{99 v_k + 2u_{k+1} + 2u_k}{101}$$

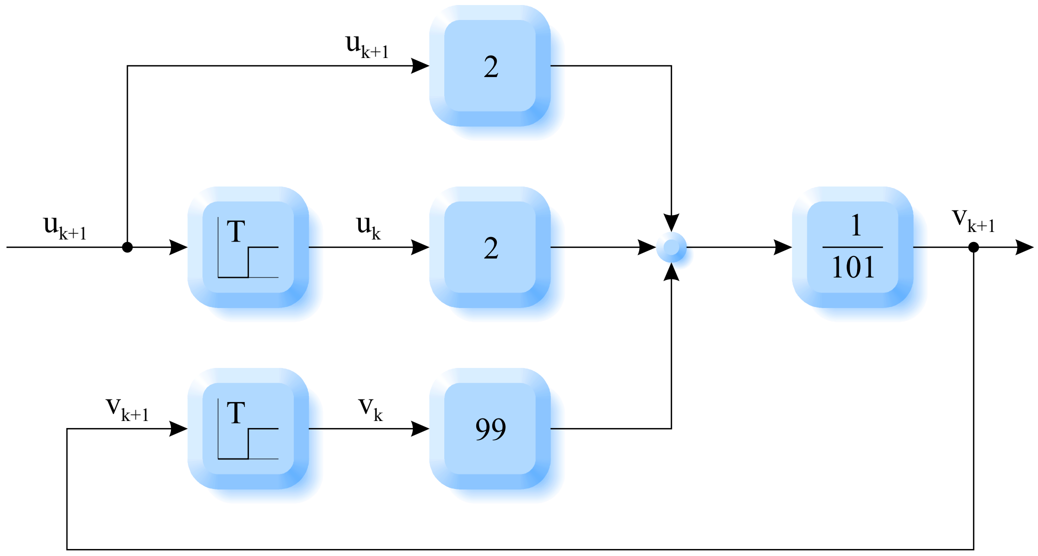


Figure 3.23: Discrete implementation of a digital low pass

Part II

Flight Control

Chapter 4

Introduction

4.1 Aviation terminology

- all designations according to [1] - [2] (extensions: V_A, Ω_K)
- all axis systems are right-handed ("right hand rule").
- summary and individual test under [3]
- many illustrations are reproduced (with the kind permission of the author) from [4].

4.1.1 Variables of motion

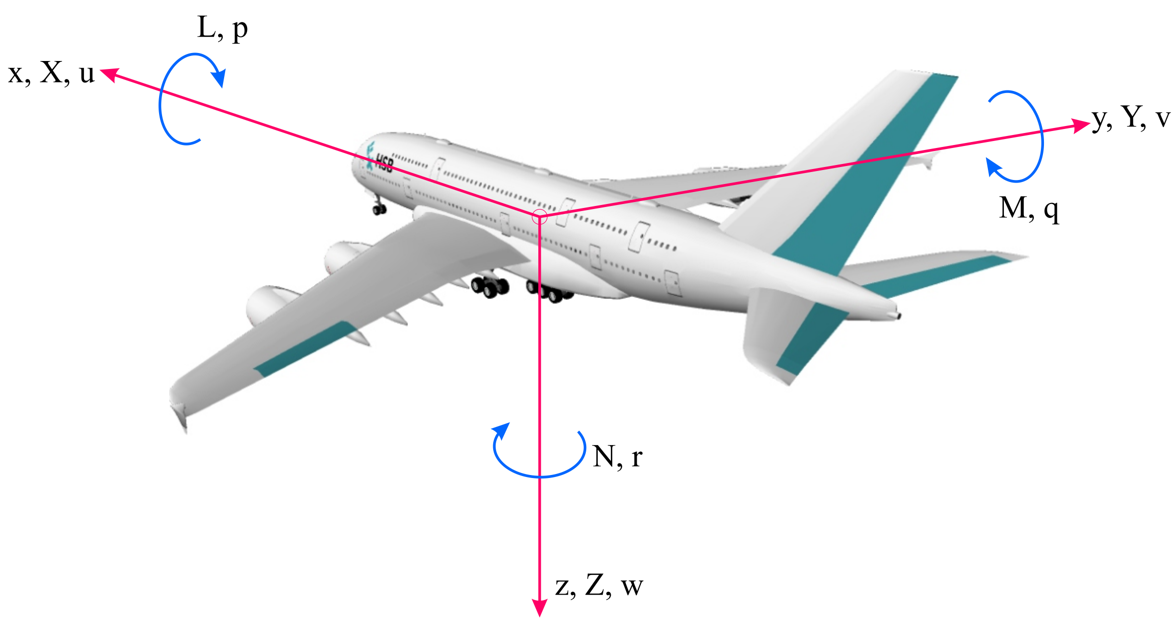


Figure 4.1: Variables of motion

Position vector (direction, distance):

$$\boldsymbol{s} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \begin{array}{c} \text{towards the front} \\ \text{towards the right} \\ \text{towards the bottom} \end{array}$$

Force vector:

$$\boldsymbol{R} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Moment vector:

$$\boldsymbol{Q} = \begin{bmatrix} L \\ M \\ N \end{bmatrix}$$

Velocity vector:

$$\boldsymbol{V} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Angular velocity vector:

$$\boldsymbol{\Omega} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \qquad \begin{array}{c} \text{roll} \\ \text{pitch} \\ \text{yaw} \end{array}$$

Attitude vector (Euler angle vector):

$$\boldsymbol{\Phi} = \begin{bmatrix} \Phi \\ \Theta \\ \Psi \end{bmatrix}$$

4.1.2 Indices

A aerodynamic

K flight-path

W wind

F thrust

f body-fixed axis system

a aerodynamic axis system

k flight-path axis system

g earth-fixed (geodetic) axis system

4.1.2.1 Examples

Resulting aerodynamic force in the aerodynamic axis system: $\boldsymbol{R}_a^A$

Resultant aerodynamic moment in the flight-path axis system: $\boldsymbol{Q}_k^A$

Thrust force in the body-fixed axis system: $\boldsymbol{F}_f^F$

Thrust moment: $\boldsymbol{Q}^F$

Weight force in the geodetic axis system: $\boldsymbol{G}_g$

Gravitational acceleration vector (acceleration due to gravity) in the body-fixed axis system: $\boldsymbol{g}_f$

4.1.3 Velocities

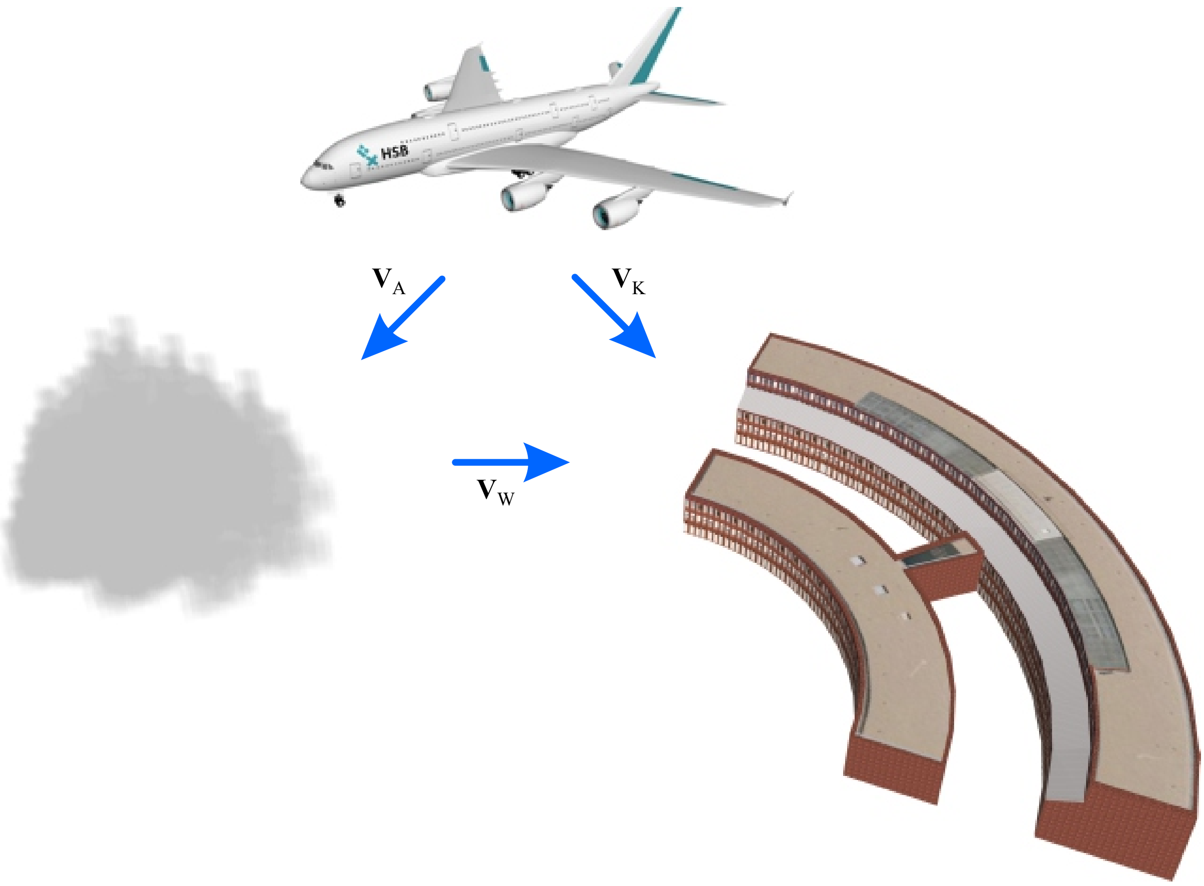


Figure 4.2: Aerodynamic velocity, flight-path velocity, and wind velocity

Flight-path velocity $\boldsymbol{V_K}$	relative velocity of the aircraft with respect to the earth
Aerodynamic velocity $\boldsymbol{V_A}$	relative velocity of the aircraft with respect to the air
Wind velocity $\boldsymbol{V_W}$	relative velocity of the air with respect to the earth

Relation between flight-path velocity, aerodynamic velocity and wind velocity vectors:

$$\boldsymbol{V_K} = \boldsymbol{V_A} + \boldsymbol{V_W}$$

Equivalent relationship for angular velocity vectors:

$$\boldsymbol{\Omega_K} = \boldsymbol{\Omega_A} + \boldsymbol{\Omega_W}$$

4.1.4 Control variables

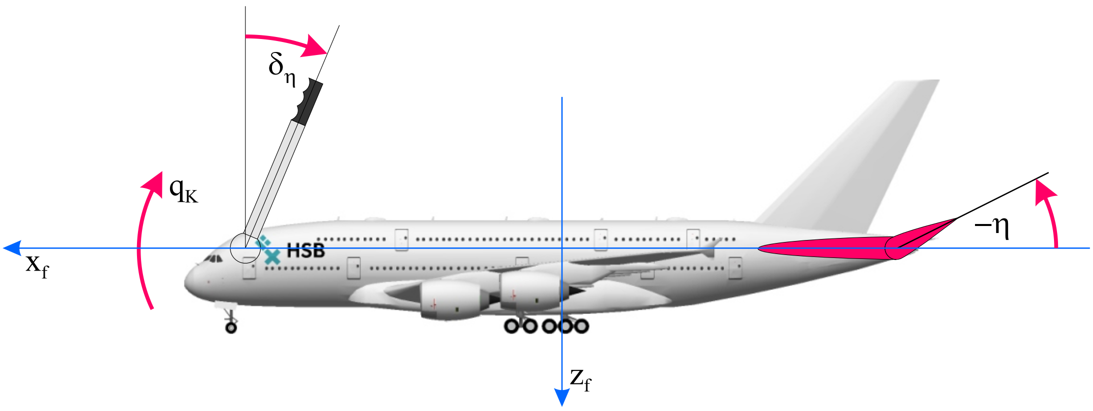


Figure 4.3: A positive stick deflection (pull) leads to a negative elevator deflection and thus to a positive pitch moment.

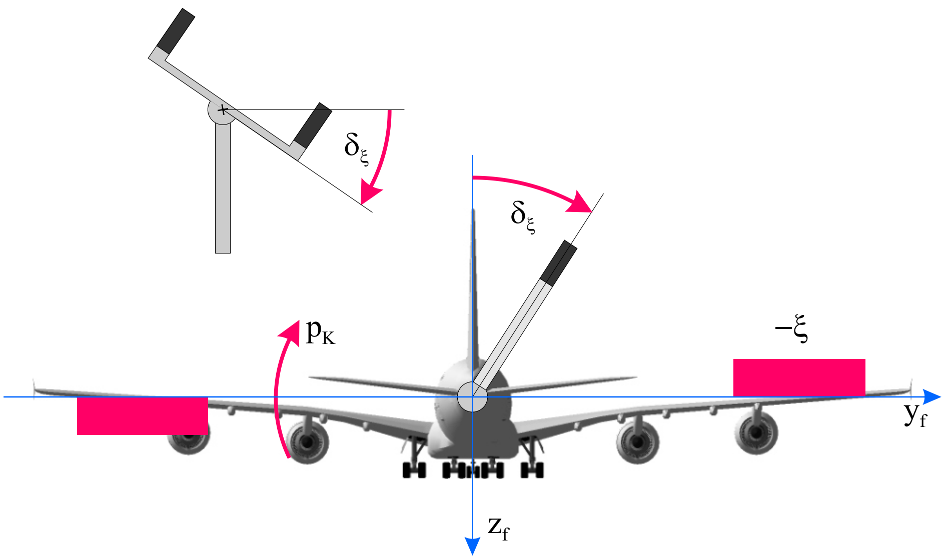


Figure 4.4: A positive stick deflection (to the right) leads to a negative aileron deflection (right aileron up) and thus to a positive roll moment.

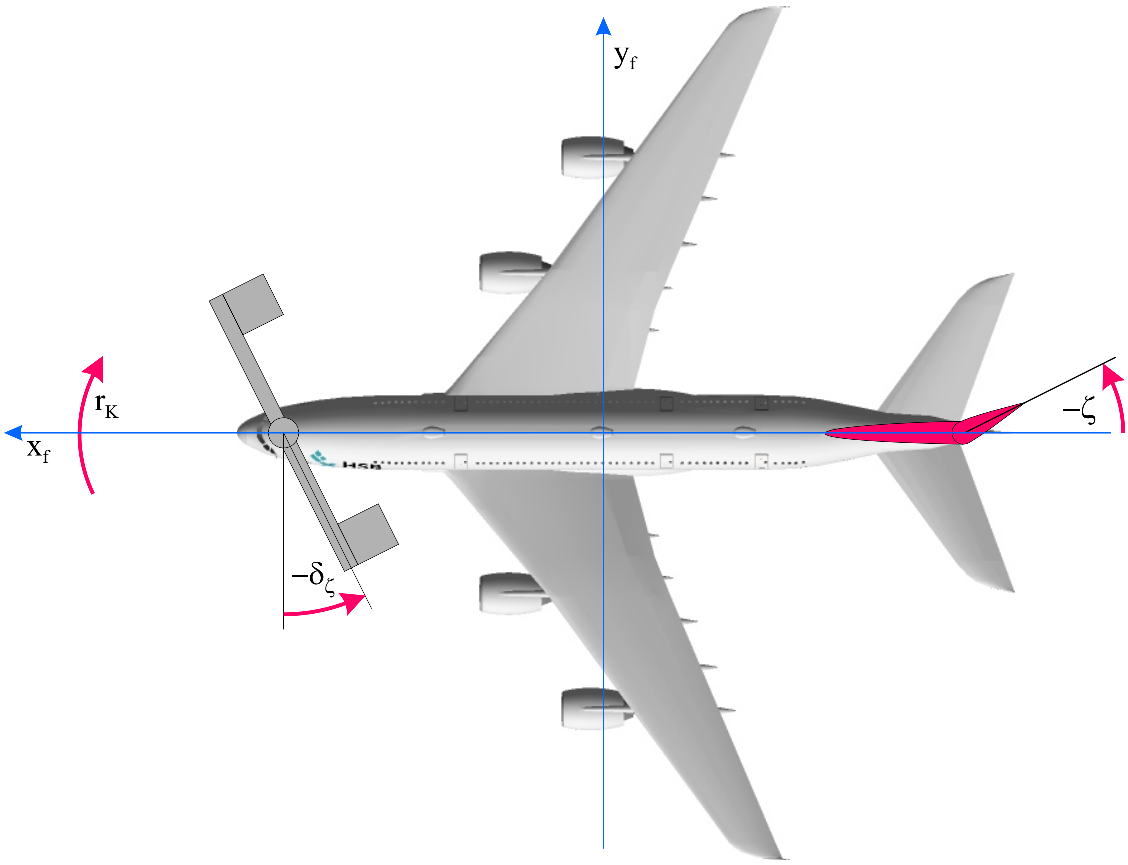


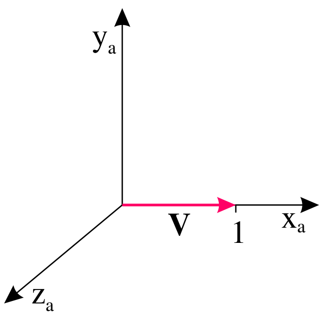
Figure 4.5: A negative pedal angle (right pedal depressed) leads to a negative rudder deflection and thus to a positive yaw moment.

4.2 Coordinate transformation

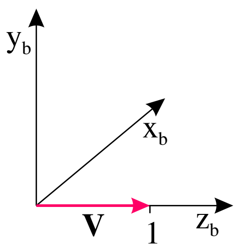
The general three-dimensional vector

$$\boldsymbol{V} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

can be described in any axis system.



(a) Vector $\boldsymbol{V}$, represented in the a axis system



(b) Same vector $\boldsymbol{V}$, represented in the b axis system

Figure 4.6: Coordinate transformation

Vector $\boldsymbol{V}$, expressed in the a axis system:

$$\boldsymbol{V_a} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}_a = \begin{bmatrix} u_a \\ v_a \\ w_a \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The b axis system is created by a 90° rotation of the a axis system about the y_a axis. The vector $\boldsymbol{V}$ is not rotated in the process.

Same vector $\boldsymbol{V}$, expressed in the b axis system:

$$\boldsymbol{V_b} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}_b = \begin{bmatrix} u_b \\ v_b \\ w_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The vector (expressed in the new axis system) now has different coordinates; however, it is still the same vector.

4.2.1 Axis systems

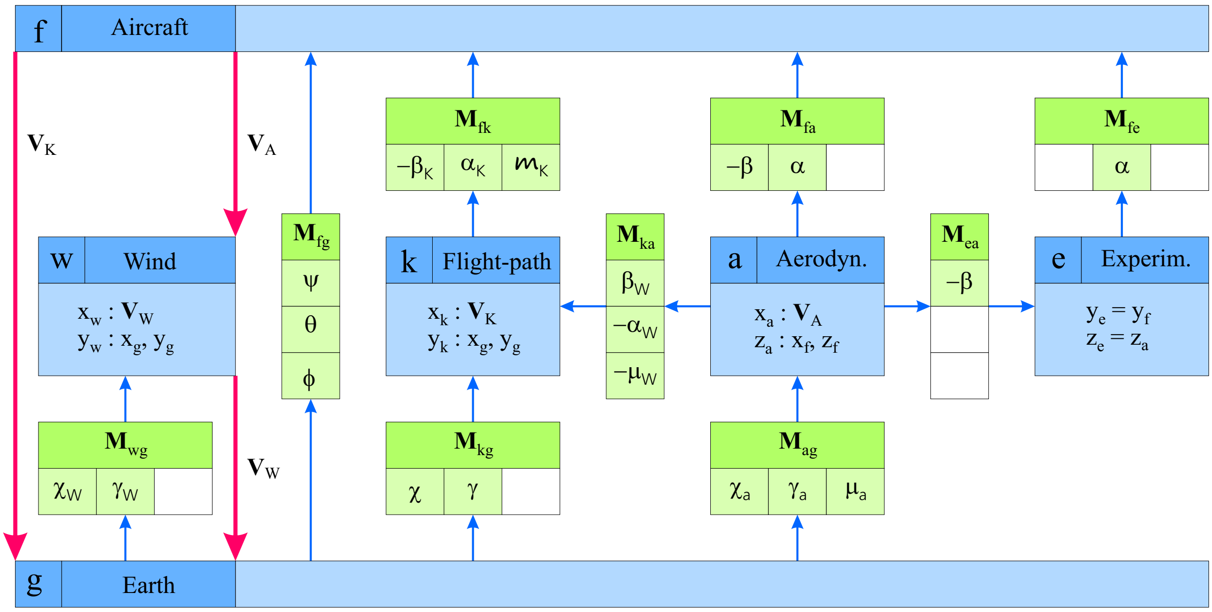


Figure 4.7: Aviation axis systems (coordinate systems) and transformation matrices (according to [1])

- The geodetic (earth-fixed) axis system (index: g) is defined by its z_g axis pointing in the direction of gravity. The x_g axis is perpendicular to the z_g axis in the earth's horizontal plane and is often assumed to be in the north direction. The y_g axis forms (as in all described axis systems) a right-handed axis system with the other two axes and therefore also lies in the earth's horizontal plane.
- The body-fixed axis system (index: f or no index) describes the attitude of the aircraft in space. The x_f axis points "forwards" (usually in the plane of symmetry

from the centre of gravity to the nose of the aircraft), the y_f axis points "to the right" (starboard) and the z_f axis points "downwards".

- The aerodynamic axis system (index: a) is defined by its x_a axis, which points in the direction of aerodynamic velocity vector $\boldsymbol{V_A}$. Since the axis system is not yet clearly defined by the definition of one axis (it could still rotate about its x_a axis), the z_a axis is defined in the plane of aircraft symmetry (x_f-z_f plane). This means that the y_f axis also lies in the x_a-y_a plane (cf. figure 4.10).
- The flight-path axis system (index: k) is defined analogously to the aerodynamic axis system: The x_k axis points in the direction of the flight-path velocity vector $\boldsymbol{V_K}$. As a second definition, the y_k axis is usually placed in the earth's horizontal plane (x_g-y_g plane) (cf. figure 4.9).

4.2.2 Rotation from the geodetic to the body-fixed axis system

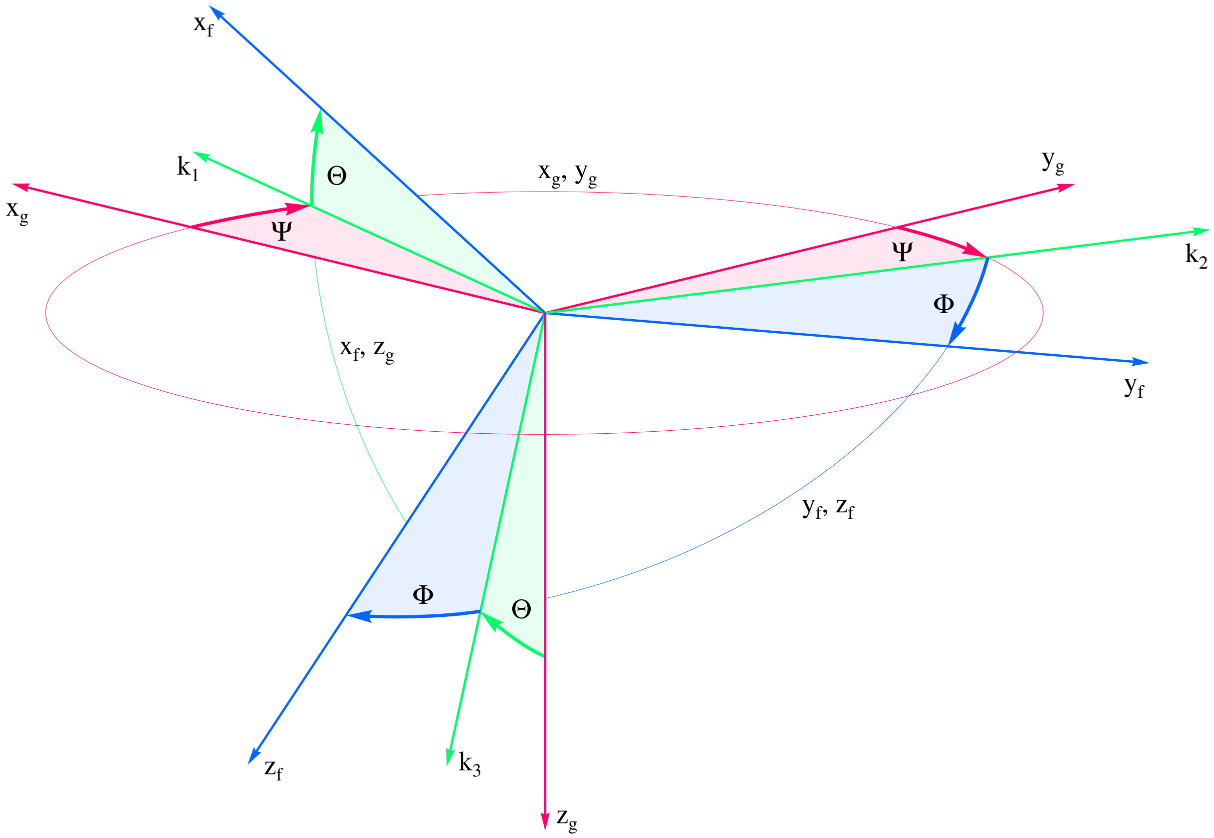


Figure 4.8: Euler angle rotation from the geodetic to the body-fixed axis system [5] (according to [1])

- The yaw angle (heading, azimuth angle) Ψ rotates in the x_g-y_g plane about the z_g axis. In doing so, the x_g axis is transformed into the nodal axis k_1 and the y_g axis into the nodal axis k_2. Main value range:: $-\pi < \Psi \leq \pi$
- The pitch angle (inclination angle) Θ rotates in the x_f-z_g plane about the k_2 axis. In the process, the k_1 axis is transformed into the x_f axis and the z_g axis into the nodal axis k_3. Main value range: $-\frac{\pi}{2} \leq \Theta \leq \frac{\pi}{2}$

- The roll angle (bank angle) Φ rotates in the y_f-z_f plane about the x_f axis. In doing so, the k_2 axis is transformed into the y_f axis and the k_3 axis into the z_f axis. Main value range: $-\pi < \Phi \leq \pi$

4.2.3 Rotation from the geodetic to the flight-path axis system

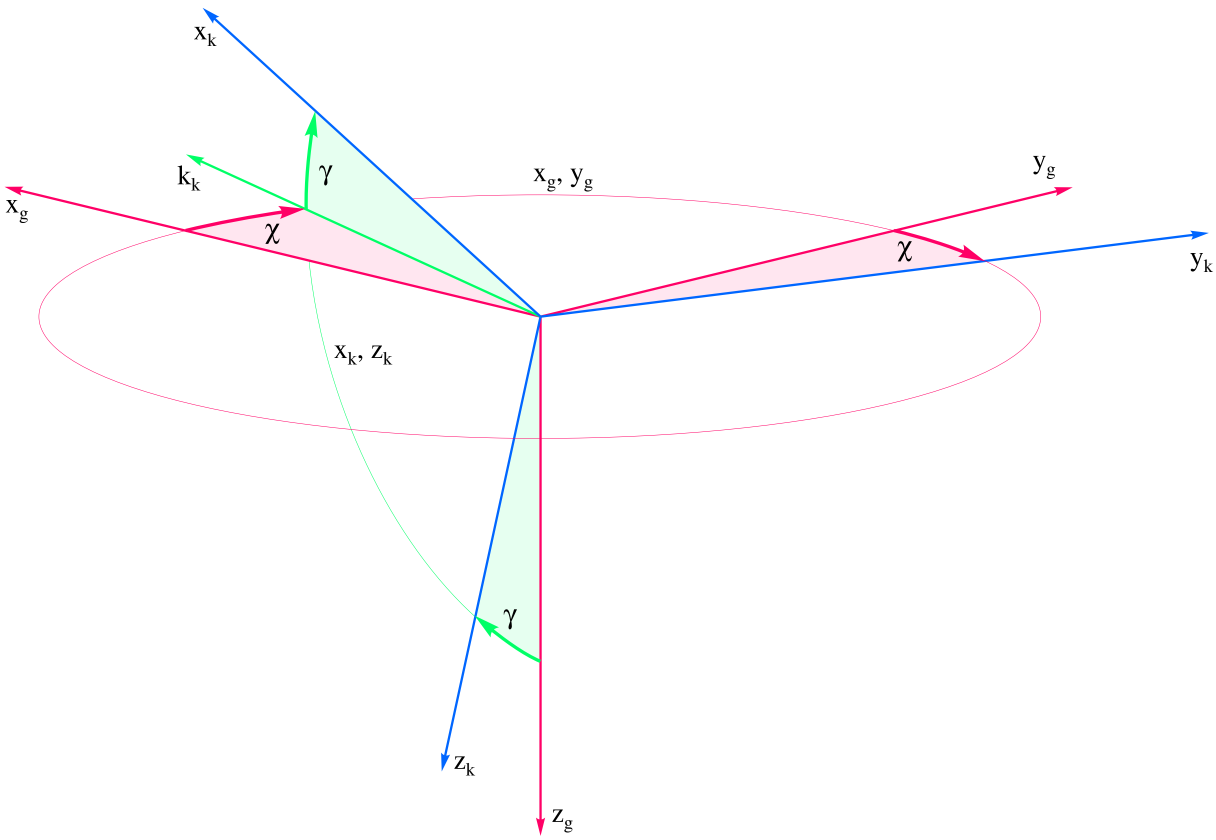


Figure 4.9: Rotation from the geodetic to the flight-path axis system [5]

- The flight-path azimuth angle χ rotates in the x_g-y_g plane about the z_g axis. In doing so, the x_g axis is transformed into the nodal axis k_k and the y_g axis into the y_k axis. Main value range: $-\pi < \chi \leq \pi$
- The flight-path angle (angle of climb, flight-path inclination angle) γ rotates in the x_k-z_k plane about the y_k axis. In the process, the k_k axis is transformed into the x_k axis and the z_g axis into the z_k axis. Main value range: $-\frac{\pi}{2} \leq \gamma \leq \frac{\pi}{2}$
- A rotation about the x_k axis (as with the Euler angles with Φ) does not take place, since the y_k axis lies in the earth's horizontal plane (x_g-y_g plane) by definition.

4.2.4 Rotation from the aerodynamic to the body-fixed system

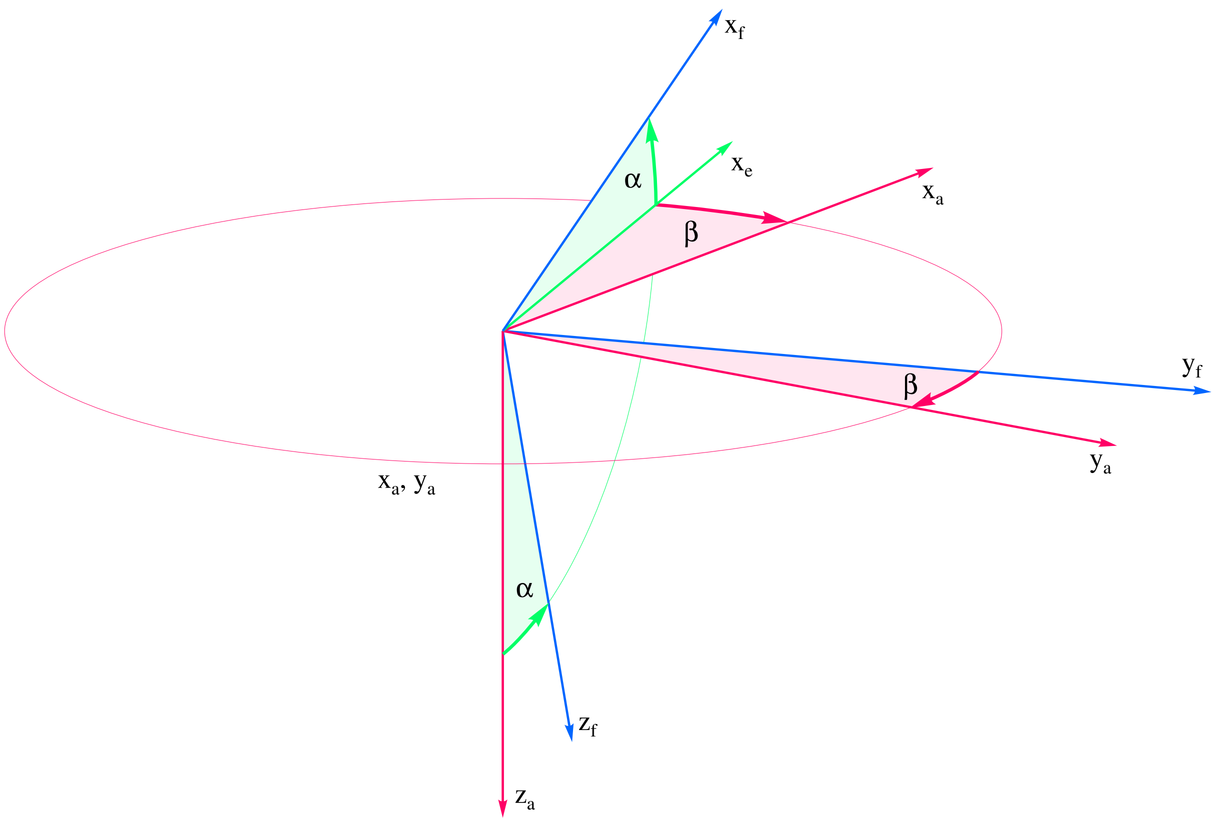


Figure 4.10: Rotation from the aerodynamic to the body-fixed system [5]

- The sideslip angle β rotates in the x_a-y_a plane about the z_a axis. By rotating in a mathematically negative direction, i. e. by "minus beta", the x_a axis is transformed into the intermediate x_e axis (experimental axis system) and the y_a axis into the y_f axis. Main value range: $-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}$
- The angle of attack α rotates in the x_f-z_f plane about the y_f axis. In the process, the x_e axis is transformed into the x_f axis and the z_a axis into the z_f axis. Main value range: $-\pi < \alpha \leq \pi$
- A rotation about the x_f axis (as with the Euler angles with Φ) does not take place, since the z_a axis lies in the plane of aircraft symmetry (x_f-z_f plane) by definition.

4.2.5 Transformation matrices

Rotation with angle w_z about a z axis:

$$\boldsymbol{M}_z = \begin{bmatrix} \cos w_z & \sin w_z & 0 \\ -\sin w_z & \cos w_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation with angle w_y about a y axis:

$$
\boldsymbol{M_y} = \begin{bmatrix} \cos w_y & 0 & -\sin w_y \\ 0 & 1 & 0 \\ \sin w_y & 0 & \cos w_y \end{bmatrix}
$$

Rotation with angle w_x about an x axis:

$$
\boldsymbol{M_x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos w_x & \sin w_x \\ 0 & -\sin w_x & \cos w_x \end{bmatrix}
$$

Total transformation matrix with a rotation sequence $w_z \to w_y \to w_x$ ("read from the right"):

$$
\begin{aligned}
\boldsymbol{M_{ges}} &= \boldsymbol{M_x} \cdot \boldsymbol{M_y} \cdot \boldsymbol{M_z} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos w_x & \sin w_x \\ 0 & -\sin w_x & \cos w_x \end{bmatrix} \begin{bmatrix} \cos w_y & 0 & -\sin w_y \\ 0 & 1 & 0 \\ \sin w_y & 0 & \cos w_y \end{bmatrix} \begin{bmatrix} \cos w_z & \sin w_z & 0 \\ -\sin w_z & \cos w_z & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Transformation from the geodetic to the body-fixed axis system:

$$
\begin{aligned}
\boldsymbol{M_{fg}} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \Phi & \sin \Phi \\ 0 & -\sin \Phi & \cos \Phi \end{bmatrix} \begin{bmatrix} \cos \Theta & 0 & -\sin \Theta \\ 0 & 1 & 0 \\ \sin \Theta & 0 & \cos \Theta \end{bmatrix} \begin{bmatrix} \cos \Psi & \sin \Psi & 0 \\ -\sin \Psi & \cos \Psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (4.1) \\
&= \begin{bmatrix} \cos \Theta \cos \Psi & \cos \Theta \sin \Psi & -\sin \Theta \\ \sin \Phi \sin \Theta \cos \Psi - \cos \Phi \sin \Psi & \sin \Phi \sin \Theta \sin \Psi + \cos \Phi \cos \Psi & \sin \Phi \cos \Theta \\ \cos \Phi \sin \Theta \cos \Psi + \sin \Phi \sin \Psi & \cos \Phi \sin \Theta \sin \Psi - \sin \Phi \cos \Psi & \cos \Phi \cos \Theta \end{bmatrix}
\end{aligned}
$$

Transformation from the aerodynamic to the body-fixed axis system:

$$
\begin{aligned}
\boldsymbol{M_{fa}} &= \begin{bmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos(-\beta) & \sin(-\beta) & 0 \\ -\sin(-\beta) & \cos(-\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \beta & -\sin \beta & 0 \\ \sin \beta & \cos \beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos \alpha \cos \beta & -\cos \alpha \sin \beta & -\sin \alpha \\ \sin \beta & \cos \beta & 0 \\ \sin \alpha \cos \beta & -\sin \alpha \sin \beta & \cos \alpha \end{bmatrix}
\end{aligned}
$$

Transformation from the geodetic to the flight-path axis system:

$$
\boldsymbol{M_{kg}} = \begin{bmatrix} \cos\gamma & 0 & -\sin\gamma \\ 0 & 1 & 0 \\ \sin\gamma & 0 & \cos\gamma \end{bmatrix} \begin{bmatrix} \cos\chi & \sin\chi & 0 \\ -\sin\chi & \cos\chi & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$
$$
= \begin{bmatrix} \cos\gamma\cos\chi & \cos\gamma\sin\chi & -\sin\gamma \\ -\sin\chi & \cos\chi & 0 \\ \sin\gamma\cos\chi & \sin\gamma\sin\chi & \cos\gamma \end{bmatrix}
$$

4.2.5.1 Transformation direction reversal

Two ways to generate the inverse transformation (reverse transformation):

1. by reversing the order of the single transformations and negative angles:
$$
\boldsymbol{M_{gk}} = \begin{bmatrix} \cos(-\chi) & \sin(-\chi) & 0 \\ -\sin(-\chi) & \cos(-\chi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(-\gamma) & 0 & -\sin(-\gamma) \\ 0 & 1 & 0 \\ \sin(-\gamma) & 0 & \cos(-\gamma) \end{bmatrix}
$$
$$
= \begin{bmatrix} \cos\chi & -\sin\chi & 0 \\ \sin\chi & \cos\chi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{bmatrix}
$$
$$
= \begin{bmatrix} \cos\chi\cos\gamma & -\sin\chi & \cos\chi\sin\gamma \\ \sin\chi\cos\gamma & \cos\chi & \sin\chi\sin\gamma \\ -\sin\gamma & 0 & \cos\gamma \end{bmatrix}
$$

2. by inverting the transformation matrix. With the rotational transformations used, inverting is simplified to transposing:

$$
\boldsymbol{M_{gk}} = \boldsymbol{M_{kg}^{-1}} = \boldsymbol{M_{kg}^{T}}
$$
$$
= \begin{bmatrix} \cos\gamma\cos\chi & \cos\gamma\sin\chi & -\sin\gamma \\ -\sin\chi & \cos\chi & 0 \\ \sin\gamma\cos\chi & \sin\gamma\sin\chi & \cos\gamma \end{bmatrix}^T
$$
$$
= \begin{bmatrix} \cos\gamma\cos\chi & -\sin\chi & \sin\gamma\cos\chi \\ \cos\gamma\sin\chi & \cos\chi & \sin\gamma\sin\chi \\ -\sin\gamma & 0 & \cos\gamma \end{bmatrix}
$$

4.2.5.2 Example

The weight vector has only a z component in the geodetic axis system, namely its magnitude:

$$
\boldsymbol{G_g} = \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix}
$$

After the transformation into the body-fixed axis system, the weight vector is dense (fully occupied):

$$
\boldsymbol{G_f} = \boldsymbol{M_{fg}} \boldsymbol{G_g} = \begin{bmatrix} \cdots & \cdots & -\sin\Theta \\ \cdots & \cdots & \sin\Phi\cos\Theta \\ \cdots & \cdots & \cos\Phi\cos\Theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix}
$$

$$
= \begin{bmatrix} -\sin\Theta \cdot mg \\ \sin\Phi\cos\Theta \cdot mg \\ \cos\Phi\cos\Theta \cdot mg \end{bmatrix} = \begin{bmatrix} -\sin\Theta \\ \sin\Phi\cos\Theta \\ \cos\Phi\cos\Theta \end{bmatrix} \cdot mg
$$

4.2.6 Conversion between Cartesian and spherical coordinates

4.2.6.1 Aerodynamic velocity conversion

The airspeed vector $\boldsymbol{V_A}$ can be expressed particularly simply in the aerodynamic axis system due to its definition. It has only a u_A component there:

$$
\boldsymbol{V_{Aa}} = \begin{bmatrix} u_A \\ v_A \\ w_A \end{bmatrix}_a = \begin{bmatrix} V_A \\ 0 \\ 0 \end{bmatrix}
$$

After the transformation into the body-fixed axis system, the relations between the Cartesian and the spherical coordinates of the aerodynamic velocity vector are obtained:

$$
\boldsymbol{V_{Af}} = \underbrace{\begin{bmatrix} u_A \\ v_A \\ w_A \end{bmatrix}_f}_{\text{Cartesian}} = \boldsymbol{M_{fa}} \boldsymbol{V_{Aa}} = \begin{bmatrix} \cos\alpha\cos\beta & \cdots & \cdots \\ \sin\beta & \cdots & \cdots \\ \sin\alpha\cos\beta & \cdots & \cdots \end{bmatrix} \begin{bmatrix} V_A \\ 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} V_A\cos\alpha\cos\beta \\ V_A\sin\beta \\ V_A\sin\alpha\cos\beta \end{bmatrix}}_{\text{spherical}}
$$

Check the identity of the norms of both representations:

$$
\begin{aligned}
|\boldsymbol{V_A}| &= \sqrt{u_{Af}^2 + v_{Af}^2 + w_{Af}^2} \\
&= \sqrt{V_A^2\cos^2\alpha\cos^2\beta + V_A^2\sin^2\beta + V_A^2\sin^2\alpha\cos^2\beta} \\
&= \sqrt{V_A^2 \underbrace{\left(\cos^2\alpha + \sin^2\alpha\right)}_{1} \cos^2\beta + V_A^2\sin^2\beta} \\
&= \sqrt{V_A^2 \underbrace{\left(\cos^2\beta + \sin^2\beta\right)}_{1}} = V_A \quad \text{q. e. d}
\end{aligned}
$$

Ratio of two Cartesian coordinates:

$$
\frac{w_{Af}}{u_{Af}} = \frac{V_A\sin\alpha\cos\beta}{V_A\cos\alpha\cos\beta} = \tan\alpha
$$

Solution for the angle of attack:

$$\alpha = \arctan\left(\frac{w_{Af}}{u_{Af}}\right)$$

Second Cartesian coordinate:

$$v_{Af} = V_A \cdot \sin\beta$$

Solution for the sideslip angle:

$$\beta = \arcsin\left(\frac{v_{Af}}{V_A}\right)$$

4.2.6.2 Conversion of the flight-path velocity

The flight-path velocity vector $\boldsymbol{V_K}$ can be expressed particularly simply in the flight-path axis system due to its definition. It has only a u_K component there:

$$\boldsymbol{V_{Kk}} = \begin{bmatrix} u_K \\ v_K \\ w_K \end{bmatrix}_k = \begin{bmatrix} V_K \\ 0 \\ 0 \end{bmatrix}$$

After the transformation into the geodetic axis system, the relationships between the Cartesian and the spherical coordinates of the flight-path velocity vector are obtained:

$$\boldsymbol{V_{Kg}} = \underbrace{\begin{bmatrix} u_K \\ v_K \\ w_K \end{bmatrix}_g}_{\text{Cartesian}} = \boldsymbol{M_{gk}}\boldsymbol{V_{Kk}} = \begin{bmatrix} \cos\gamma\cos\chi & \cdots & \cdots \\ \cos\gamma\sin\chi & \cdots & \cdots \\ -\sin\gamma & \cdots & \cdots \end{bmatrix} \begin{bmatrix} V_K \\ 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} V_K\cos\gamma\cos\chi \\ V_K\cos\gamma\sin\chi \\ -V_K\sin\gamma \end{bmatrix}}_{\text{spherical}}$$

Ratio of two Cartesian coordinates:

$$\frac{v_{Kg}}{u_{Kg}} = \frac{V_K\cos\gamma\sin\chi}{V_K\cos\gamma\cos\chi} = \tan\chi$$

Solution for the flight-path azimuth:

$$\chi = \arctan\left(\frac{v_{Kg}}{u_{Kg}}\right)$$

Third Cartesian coordinate:

$$w_{Kg} = -V_K\sin\gamma$$

Solution for the angle of climb:

$$\gamma = -\arcsin\left(\frac{w_{Kg}}{V_K}\right)$$

4.2.6.3 Summary of the conversions

Spherical $\rightarrow$ Cartesian:

$$\begin{aligned}
u_{Af} &= V_A \cos\alpha \cos\beta \\
v_{Af} &= V_A \sin\beta \\
w_{Af} &= V_A \sin\alpha \cos\beta \\
u_{Kg} &= V_K \cos\gamma \cos\chi \\
v_{Kg} &= V_K \cos\gamma \sin\chi \\
w_{Kg} &= -V_K \sin\gamma
\end{aligned}$$

Cartesian $\rightarrow$ spherical:

$$\begin{aligned}
V_A &= \sqrt{u_{Af}^2 + v_{Af}^2 + w_{Af}^2} \\
\alpha &= \arctan\left(\frac{w_{Af}}{u_{Af}}\right) \\
\beta &= \arcsin\left(\frac{v_{Af}}{V_A}\right) \\
V_K &= \sqrt{u_{Kg}^2 + v_{Kg}^2 + w_{Kg}^2} \\
\gamma &= -\arcsin\left(\frac{w_{Kg}}{V_K}\right) \\
\chi &= \arctan\left(\frac{v_{Kg}}{u_{Kg}}\right)
\end{aligned}$$

4.2.7 Representation of angles and vectors

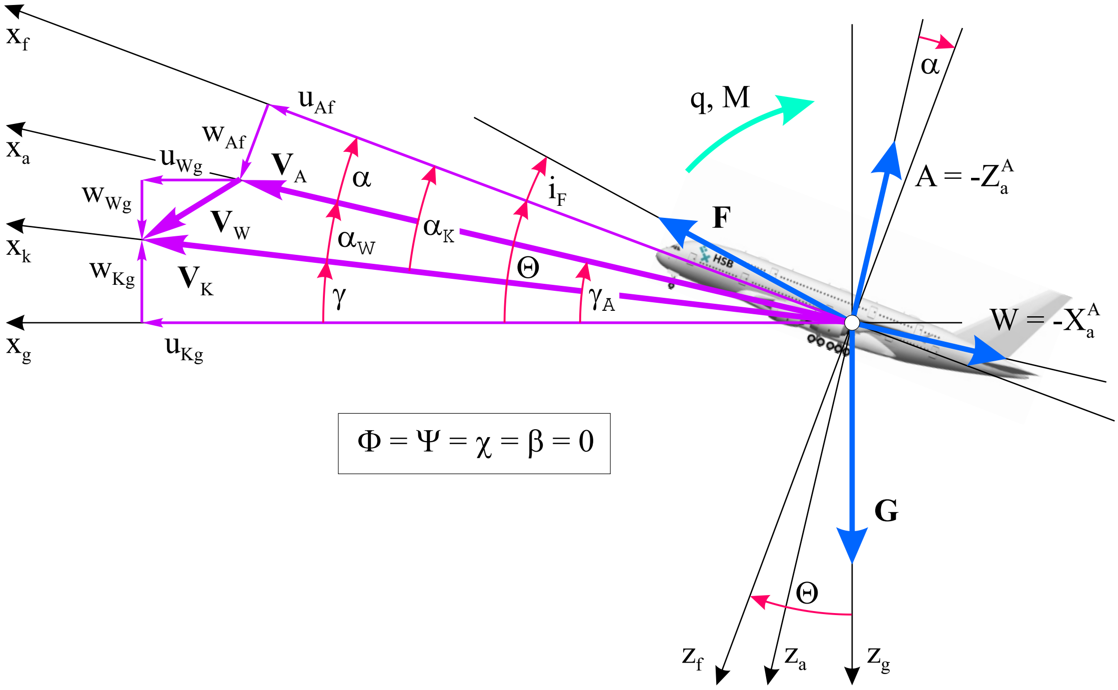


Figure 4.11: Angles and vectors in the x-z plane

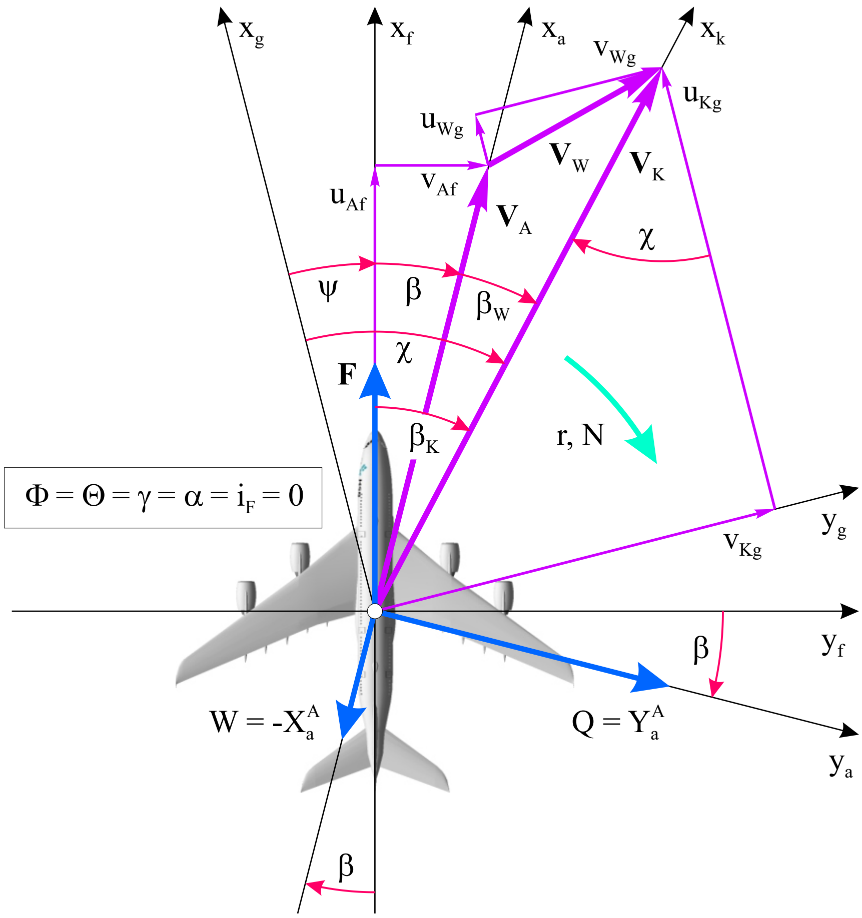


Figure 4.12: Angles and vectors in the x-y plane

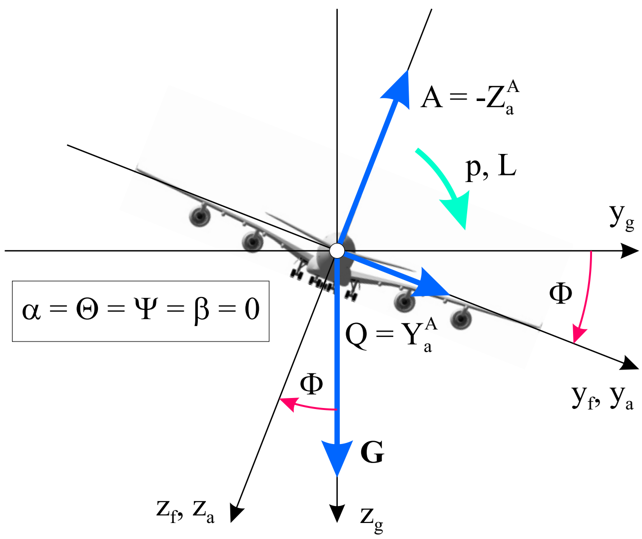


Figure 4.13: Angles and vectors in the y-z plane

Chapter 5

Subsystems

5.1 Aerodynamics

The relative velocity V_A between the aircraft and the air with density ρ creates dynamic pressure:

$$\bar{q} = \frac{\rho}{2} V_A^2$$

The product of dynamic pressure $\bar{q}$ and reference wing area S is called aerodynamic force unit E:

$$E = \bar{q} \cdot S$$

The aerodynamic forces result as the product of the aerodynamic force unit with the dimensionless coefficients:

Lift:

$$A = E \cdot C_A$$

Drag:

$$W = E \cdot C_W$$

Side force:

$$Q = E \cdot C_Q$$

For the moments, a reference length is also required for dimensional reasons. Usually, the mean aerodynamic chord l_μ is used today for all moments:

Roll moment:

$$L = E \cdot l_\mu \cdot C_l$$

Pitch moment:

$$M = E \cdot l_\mu \cdot C_m$$

Yaw moment:

$$N = E \cdot l_\mu \cdot C_n$$

5.1.1 Coefficients

The coefficients are non-linear functions of the respective aerodynamic parameters:

5.1.1.1 Coefficients of the longitudinal motion

Lift coefficient:

$$C_A = C_A\left(\alpha,\, \eta,\, Ma,\, q,\, \dot{\alpha},\, \ldots\right)$$

Drag coefficient:

$$C_W = C_W\left(\alpha,\, \eta,\, Ma,\, \ldots\right)$$

Pitch moment coefficient:

$$C_m = C_m\left(\alpha,\, \eta,\, Ma,\, q,\, \dot{\alpha},\, \ldots\right)$$

Alternative modeling of the drag coefficient via the drag polar:

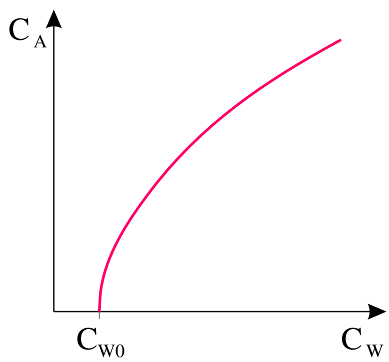


Figure 5.1: Drag polar

Drag coefficient:

$$C_W = C_{W0} + k \cdot C_A^2$$

where C_{W0} is the zero drag (no lift) and $k \cdot C_A^2$ is the lift induced drag.

5.1.1.2 Coefficients of lateral motion

Side force coefficient:

$$C_Q = C_Q\,(\beta,\, p,\, r,\, \xi,\, \zeta,\, \ldots)$$

Roll moment coefficient:

$$C_l = C_l\,(\beta,\, p,\, r,\, \xi,\, \zeta,\, \ldots)$$

Yaw moment coefficient:

$$C_n = C_n\,(\beta,\, p,\, r,\, \xi,\, \zeta,\, \ldots)$$

5.1.2 Linear derivative aerodynamics

An aerodynamic derivative is the partial derivative of an aerodynamic coefficient with respect to an aerodynamic influence quantity.

5.1.2.1 Example: lift characteristic

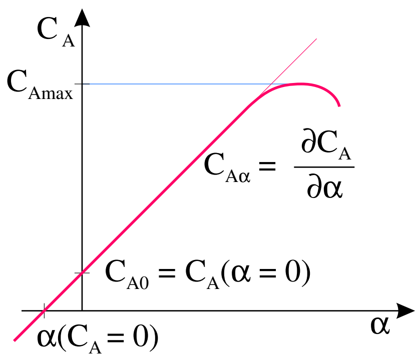


Figure 5.2: Lift characteristic (lift coefficient over angle of attack)

Within a working range (in the vicinity of a working point) a linear dependence of the lift coefficient on the angle of attack is assumed. There, the slope of the characteristic curve is constant and corresponds to the derivative $C_{A\alpha}$:

Lift due to angle of attack:

$$C_{A\alpha} = \frac{\partial C_A}{\partial \alpha}$$

Analogously, further lift derivatives are defined:

Lift due to elevator deflection:

$$C_{A\eta} = \frac{\partial C_A}{\partial \eta}$$

Lift due to Mach number::

$$C_{AMa} = \frac{\partial C_A}{\partial Ma}$$

Since the derivatives (just like the coefficients) are dimensionless, an angular speed (unit: s^{-1}) must first be rendered dimensionless (normalised) with a reference time constant before it can be partially derived. Usually, the time constant

$$T_N = \frac{l_\mu}{V_A}$$

is used for normalisation:

Normalised aerodynamic pitch speed:

$$q_A^* = T_N \cdot q_A = \frac{l_\mu}{V_A} \cdot q_A$$

The corresponding derivative is obtained by partial derivation with respect to the normalised angular speed:

Lift due to pitch speed:

$$C_{Aq} = \frac{\partial C_A}{\partial \left(q_A^*\right)}$$

The total lift coefficient, within the framework of the described linear derivative aerodynamics, is composed of the linear combination of the individual influences:

Total lift coefficient:

$$C_A = C_{A_0} + C_{A\alpha} \cdot \alpha + C_{A\eta} \cdot \eta + C_{AMa} \cdot Ma + C_{Aq} \cdot q_A^* + \dots$$

The same applies to the other force and moment coefficients:

Total pitch moment coefficient:

$$C_m = C_{m_0} + C_{m\alpha} \cdot \alpha + C_{m\eta} \cdot \eta + C_{mMa} \cdot Ma + C_{mq} \cdot q_A^* + \dots$$

Total side force coefficient:

$$C_Q = C_{Q\beta} \cdot \beta + C_{Qp} \cdot p_A^* + C_{Qr} \cdot r_A^* + C_{Q\xi} \cdot \xi + C_{Q\zeta} \cdot \zeta + \dots$$

Total roll moment coefficient:

$$C_l = C_{l\beta} \cdot \beta + C_{lp} \cdot p_A^* + C_{lr} \cdot r_A^* + C_{l\xi} \cdot \xi + C_{l\zeta} \cdot \zeta + \dots$$

Total yaw moment coefficient:

$$C_n = C_{n\beta} \cdot \beta + C_{np} \cdot p_A^* + C_{nr} \cdot r_A^* + C_{n\xi} \cdot \xi + C_{n\zeta} \cdot \zeta + \dots$$

The individual derivatives are usually designated according to their cause-effect relationship:

Pitch damping (damping of the pitch motion):

$$C_{mq}$$

Yaw damping (damping of the yaw motion):

$$C_{nr}$$

Wind vane stability (alignment "into the wind"):

$$C_{n\beta}$$

Sideslip roll moment (roll moment due to sideslip):

$$C_{l\beta}$$

Yaw side force ("Side force due to yaw"):

$$C_{Qr}$$

etc.

5.2 Engine

Thrust vector (maximum thrust) dependent on

- air inlet and outlet velocity vectors (thrust vector angle, angle of attack, sideslip angle)
- air density (altitude)
- Mach number
- ...

Low pass behaviour:

$$T_F \cdot \dot{F} + F = F_c$$

with

T_F engine time constant

F thrust

F_c thrust reference

Thrust moment vector:

$$\boldsymbol{Q_F} = \boldsymbol{r_F} \times \boldsymbol{F_F} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} \times \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$$

with

$\boldsymbol{F_F}$ thrust vector

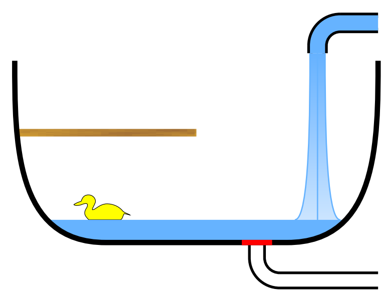
$\boldsymbol{r_F}$ thrust vector application point (distance of the engine from the reference point)

$\boldsymbol{Q_F}$ thrust moment vector

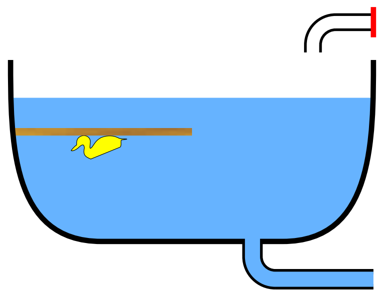
5.3 Actuator dynamics

5.3.1 Animal experiment

Suppose there was a board halfway up your bathtub with a plastic duck floating on the water underneath.



(a) Drain closed, inlet opened, water level rises, duck rises.



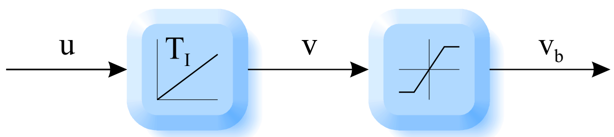
(b) Drain opened, inlet closed, water level drops, duck does not sink.

Figure 5.3: Animal experiment

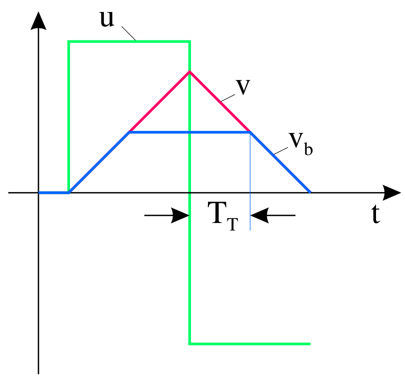
- As long as the inlet is opened, the outlet is closed, and the duck is not yet touching the board, the duck will rise together with the rising water level.
- As soon as the duck hits the board, it stops at a constant height. The water level continues to rise regardless.
- When the inlet is closed and the outlet is opened, the water level begins to sink. The duck, however, does not sink yet.
- Only when the sinking water level reaches the duck under the board can it sink together with the water.

5.3.2 Generalisation

Bathtub	energy storage, integrator, dynamic system
Inlet or outlet	input variable u of the integrator
Water level	energy content, state, output variable v of the integrator
Board	saturation v_b of the output variable



(a) Block diagram



(b) Time plots

Figure 5.4: Subsequently limited integrator

Problem: If the output variable of a dynamic system is limited, it can happen that the internal state variables "run full" and the reaction of the system only becomes visible after an undesired time delay T_T, although the limited output signal should actually react immediately.

Solution: Additionally stop the corresponding state variables when the output variable runs into its saturation. In the example: Stop the integrator by explicitly setting its input variable to zero: Close the inlet.

Notice:

Never thoughtlessly limit the output of a dynamic system.

5.4 Wind

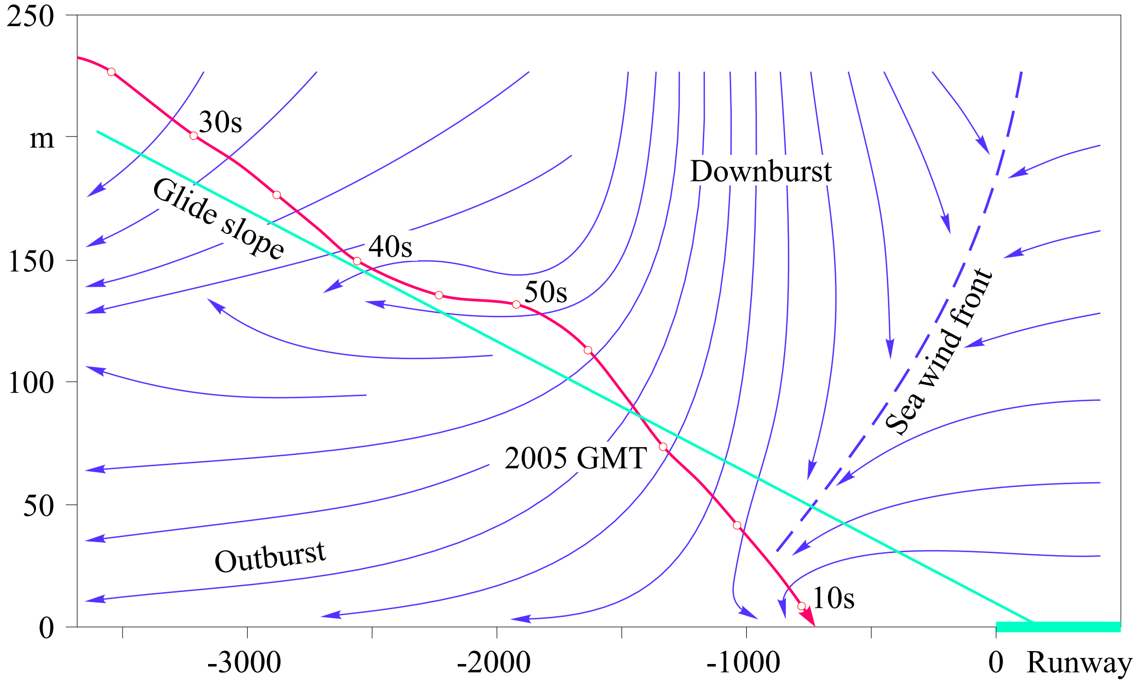


Figure 5.5: Loss of a passenger aircraft at J. F. Kennedy Airport on 24. 6. 1975

The total wind can be composed of three parts (stationary wind, turbulence (gusts) and wind shear):

$$\boldsymbol{V_W} = \boldsymbol{V_{W\,Stat}} + \boldsymbol{V_{W\,Turb}} + \boldsymbol{V_{W\,Shear}}$$

5.4.1 Turbulence

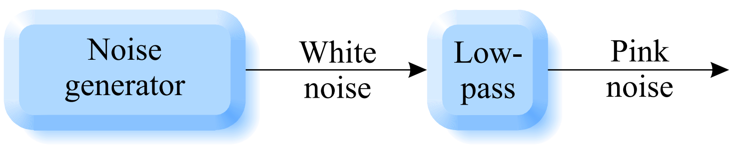


Figure 5.6: White noise: equal power density for all frequencies. Pink noise: high frequencies have lower power density.

5.4.2 Wind gradients, wind shear

5.4.2.1 The Nabla operator

Nabla operator (partial derivative operator):

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix}^T$$

Applying the Nabla operator to a scalar field p yields a vector (the gradient):

Gradient:

$$\nabla p = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} p = \begin{bmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \\ \frac{\partial p}{\partial z} \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \operatorname{grad}(p)$$

The application of the Nabla operator to a vector field $\boldsymbol{V}$ yields – depending on the type of the product – a scalar (divergence), a vector (rotation) or a matrix (Jacobian matrix):

Divergence (scalar product, inner product):

$$\nabla \cdot \boldsymbol{V} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \boldsymbol{V} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = u_x + v_y + w_z = \operatorname{div}(\boldsymbol{V})$$

Rotation (cross product):

$$\nabla \times \boldsymbol{V} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \boldsymbol{V} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{bmatrix} = \begin{bmatrix} w_y - v_z \\ u_z - w_x \\ v_x - u_y \end{bmatrix} = \operatorname{rot}(\boldsymbol{V})$$

Jacobian (dyadic product, outer product):

$$\begin{aligned} \nabla \cdot \boldsymbol{V^T} &= \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \boldsymbol{V^T} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}^T = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} u & v & w \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{bmatrix} \end{aligned}$$

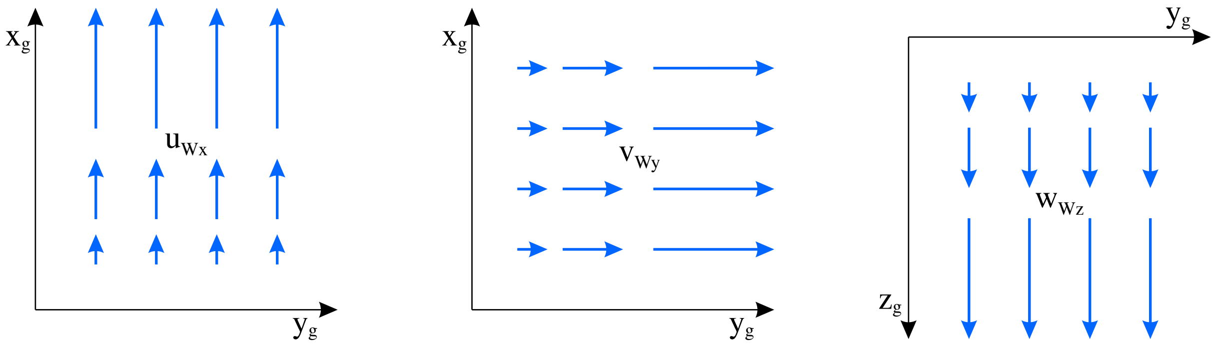
Source-freeness:

$$\operatorname{div}(\boldsymbol{V}) = u_x + v_y + w_z = 0$$

Rotation-freeness (irrotationality):

$$\operatorname{rot}(\boldsymbol{V}) = \begin{bmatrix} w_y - v_z \\ u_z - w_x \\ v_x - u_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

5.4.2.2 Wind gradients



(a) u_{Wx} (wind in x direction increasing in x direction)

(b) v_{Wy} (wind in y direction increasing in y direction)

(c) w_{Wz} (wind in z direction increasing in z direction)

Figure 5.7: Wind gradients

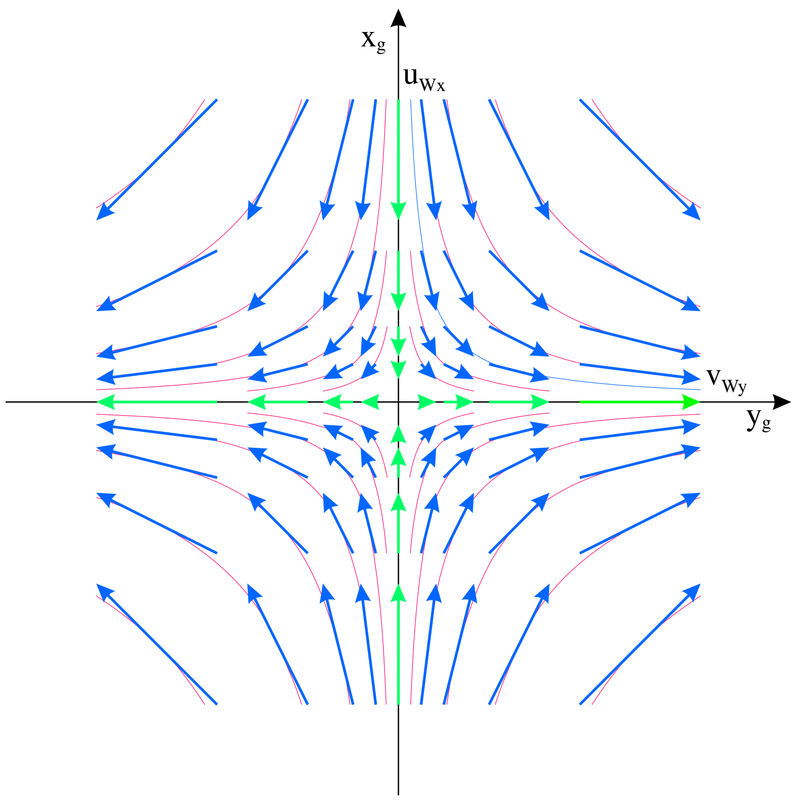


Figure 5.8: Horizontal, source-free wind field ($u_{Wx} = -v_{Wy}$)

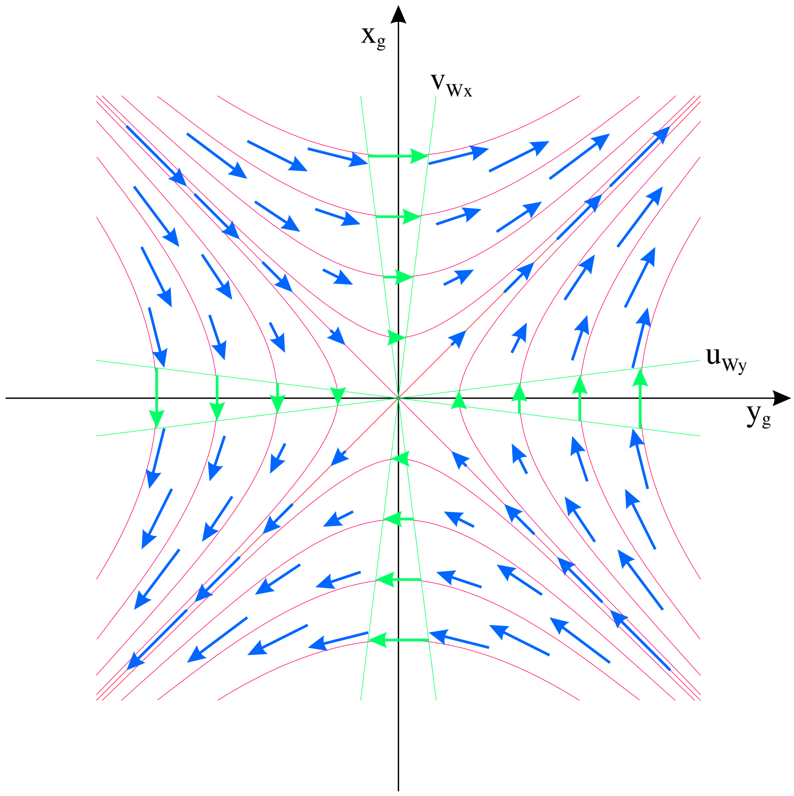


Figure 5.9: Horizontal, rotation-free wind field ($u_{Wy} = v_{Wx}$)

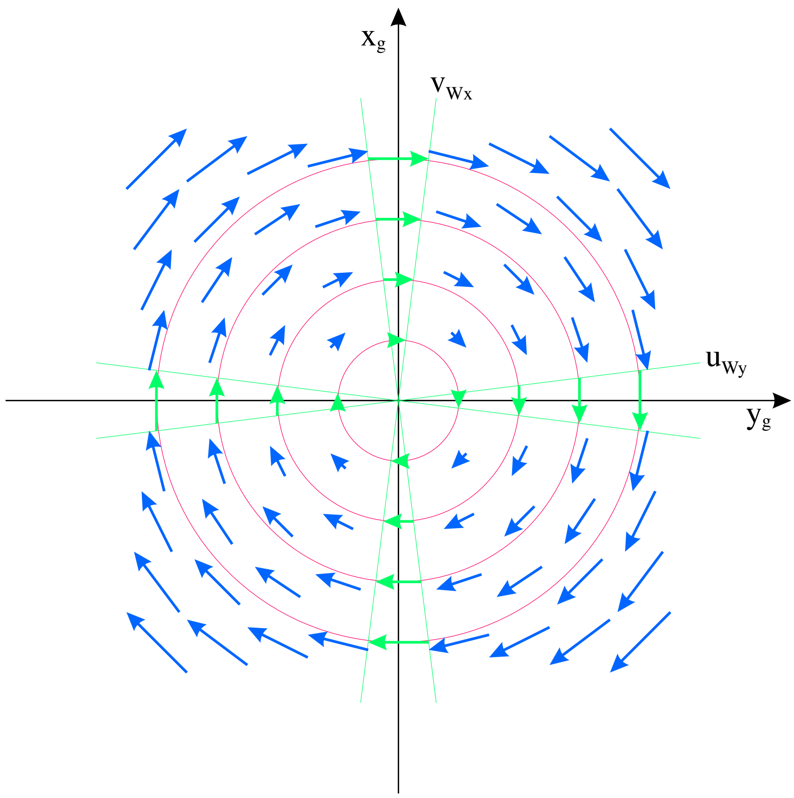


Figure 5.10: Horizontal, rotating wind field ($u_{Wy} = -v_{Wx}$)

Description of the wind component caused by wind shear using the Jacobian matrix (shear tensor):

$$\boldsymbol{V_{W\,Shear}} = \begin{bmatrix} u_W \\ v_W \\ w_W \end{bmatrix}_{Shear} = \left(\nabla \cdot \boldsymbol{V_W^T}\right)^T \cdot \boldsymbol{s} = \begin{bmatrix} u_{Wx} & v_{Wx} & w_{Wx} \\ u_{Wy} & v_{Wy} & w_{Wy} \\ u_{Wz} & v_{Wz} & w_{Wz} \end{bmatrix}^T \boldsymbol{s}$$
$$= \begin{bmatrix} u_{Wx} & u_{Wy} & u_{Wz} \\ v_{Wx} & v_{Wy} & v_{Wz} \\ w_{Wx} & w_{Wy} & w_{Wz} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u_{Wx} \cdot x + u_{Wy} \cdot y + u_{Wz} \cdot z \\ v_{Wx} \cdot x + v_{Wy} \cdot y + v_{Wz} \cdot z \\ w_{Wx} \cdot x + w_{Wy} \cdot y + w_{Wz} \cdot z \end{bmatrix}$$

5.4.3 Flight in a stationary wind field

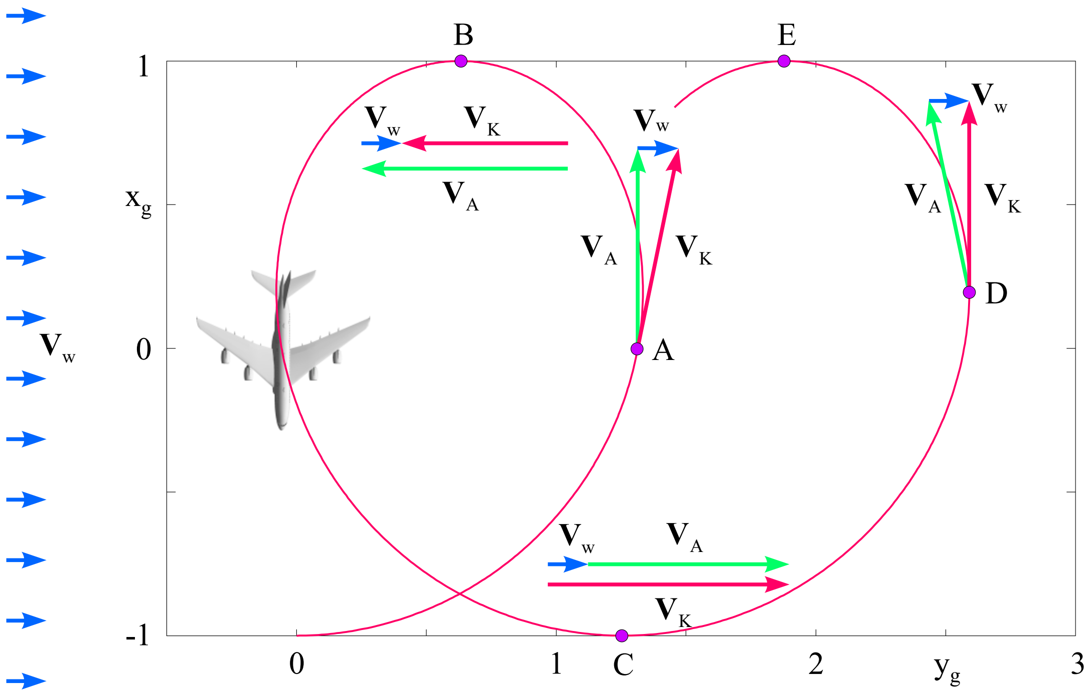


Figure 5.11: Flight in a stationary wind field (trochoid, cycloid)

On every point:

$$\boldsymbol{V_K} = \boldsymbol{V_A} + \boldsymbol{V_W} \quad \text{(vector sum)}$$

The aircraft flies a stationary, horizontal turn without sideslip angle relative to the air. All aerodynamic variables (angle of attack, lift, . . .) are constant.

The flight-path velocity vector is always oriented tangentially to the trajectory.

Point A "reversal point" of the "circle" with respect to the air, at $x_g = 0$. The aerodynamic velocity vector points exactly in the x_g direction.

Point B the aircraft flies exactly in negative y_g direction, against the wind. The magnitude (norm) of the flight-path velocity vector (flight-path speed) is minimum.

Point C the aircraft flies exactly in positive y_g direction, with the wind. The magnitude of the flight-path velocity vector is maximum.

Point D reversal point of the trochoid with respect to the Earth. The flight-path velocity vector points exactly in the x_g direction.

5.4.3.1 Flight-path speed and energy

Between points B and C, the magnitude of the flight-path velocity vector increases; the aircraft thus accelerates relative to the earth, which is assumed to be at rest. Between C and E the aircraft decelerates.

With respect to the air, the aircraft flies a circle. The aerodynamic velocity vector therefore rotates with the constant yaw angle derivative:

$$\boldsymbol{V_{Ag}} = \begin{bmatrix} V_A \sin\left(\dot{\Psi}t\right) \\ V_A \cos\left(\dot{\Psi}t\right) \end{bmatrix}$$

The wind comes from the west and therefore has only a y_g component:

$$\boldsymbol{V_{Wg}} = \begin{bmatrix} 0 \\ V_W \end{bmatrix}$$

The flight-path velocity vector in the earth-fixed axis system results from the vector sum:

$$\boldsymbol{V_{Kg}} = \boldsymbol{V_{Ag}} + \boldsymbol{V_{Wg}} = \begin{bmatrix} V_A \sin\left(\dot{\Psi}t\right) \\ V_A \cos\left(\dot{\Psi}t\right) \end{bmatrix} + \begin{bmatrix} 0 \\ V_W \end{bmatrix} = \begin{bmatrix} V_A \sin\left(\dot{\Psi}t\right) \\ V_A \cos\left(\dot{\Psi}t\right) + V_W \end{bmatrix}$$

The magnitude of the flight-path velocity is a periodic function of time:

$$\begin{aligned} V_{Kg} = |\boldsymbol{V_{Kg}}| &= \sqrt{\left(V_A \sin\left(\dot{\Psi}t\right)\right)^2 + \left(V_W + V_A \cos\left(\dot{\Psi}t\right)\right)^2} \\ &= \sqrt{V_A^2 \sin^2\left(\dot{\Psi}t\right) + V_W^2 + 2V_W V_A \cos\left(\dot{\Psi}t\right) + V_A^2 \cos^2\left(\dot{\Psi}t\right)} \\ &= \sqrt{V_A^2 + V_W^2 + 2V_W V_A \cos\left(\dot{\Psi}t\right)} \end{aligned}$$

If the flight-path speed of the aircraft changes in the course of the trochoid, the kinetic energy cannot be constant either:

$$E_{kin} = \frac{1}{2} m V_{Kg}^2 \neq const.$$

However, since the potential energy of the altitude remains constant during horizontal curved flight, the energy must be taken directly from the surrounding wind field or given off to it.

5.5 Kinetics

The aircraft motion has six degrees of freedom:

Three **translational** degrees of freedom:

- forward/backward
- right/left
- up/down

Three **rotational** degrees of freedom:

- roll (about the x axis)
- pitch (about the y axis)
- yaw (about the z axis)

Each degree of freedom is described by two states (speed and position, or angular speed and attitude angle) $\rightarrow$ 12 states in total.

These 12 states can be combined into four three-dimensional state vectors:

Angular velocity vector:

$$\boldsymbol{\Omega_K} = \begin{bmatrix} p_K \\ q_K \\ r_K \end{bmatrix}$$

Attitude vector:

$$\boldsymbol{\Phi} = \begin{bmatrix} \Phi \\ \Theta \\ \Psi \end{bmatrix}$$

Flight-path velocity vector:

$$\boldsymbol{V_K} = \begin{bmatrix} u_K \\ v_K \\ w_K \end{bmatrix}$$

Position vector:

$$\boldsymbol{s} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The entire kinetics is then described by a system of four coupled, non-linear vector differential equations

$$
\begin{aligned}
\dot{\boldsymbol{\Omega}}_{\boldsymbol{K}} &= \boldsymbol{f}_{\Omega}\left(\boldsymbol{Q},\ \boldsymbol{\Omega}_{\boldsymbol{K}}\right) \\
\dot{\boldsymbol{\Phi}} &= \boldsymbol{f}_{\boldsymbol{\Phi}}\left(\boldsymbol{\Omega}_{\boldsymbol{K}},\ \boldsymbol{\Phi}\right) \\
\dot{\boldsymbol{V}}_{\boldsymbol{K}} &= \boldsymbol{f}_{\boldsymbol{V}}\left(\boldsymbol{R},\ \boldsymbol{V}_{\boldsymbol{K}},\ \boldsymbol{\Omega}_{\boldsymbol{K}},\ \boldsymbol{\Phi}\right) \\
\dot{\boldsymbol{s}} &= \boldsymbol{f}_{\boldsymbol{s}}\left(\boldsymbol{V}_{\boldsymbol{K}},\ \boldsymbol{\Phi}\right)
\end{aligned}
$$

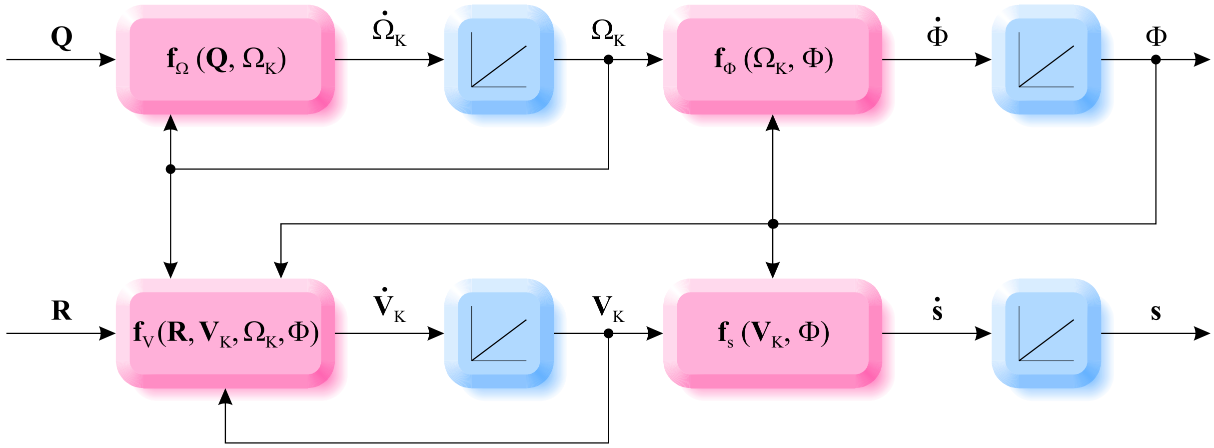


Figure 5.12: General kinetics of aircraft motion (6 degrees of freedom)

5.5.1 Differential equation of the position vector

"Velocity is the change in position over time":

$$\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t} = \boldsymbol{V_K}$$

Expressed in the earth-fixed axis system:

$$\frac{\mathrm{d}\boldsymbol{s_g}}{\mathrm{d}t} = \boldsymbol{V_{Kg}}$$

Transformation of the flight-path velocity:

$$\frac{\mathrm{d}\boldsymbol{s_g}}{\mathrm{d}t} = \boldsymbol{M_{gf}}\boldsymbol{V_{Kf}}$$

Use the derivative point for the direct derivative in the geodetic (inertial) axis system:

$$\dot{\boldsymbol{s}}_{\boldsymbol{g}} = \boldsymbol{M_{gf}}\boldsymbol{V_{Kf}}$$

5.5.2 Differential equation of the attitude vector

"The angular velocity is the change in the attitude over time":

$$\frac{\mathrm{d}\boldsymbol{\Phi}}{\mathrm{d}t} = \boldsymbol{\Omega_K}$$

Expressed in the body-fixed axis system:

$$\left(\frac{\mathrm{d}\boldsymbol{\Phi}}{\mathrm{d}t}\right)_f = \boldsymbol{\Omega_{Kf}}$$

If now all angle components of the attitude vector were to rotate about the body-fixed axes, *then*

$$\left(\frac{\mathrm{d}\boldsymbol{\Phi}}{\mathrm{d}t}\right)_f$$

would contain the time derivatives of the Euler angles

$$\begin{bmatrix}\dot{\Phi}\\ \dot{\Theta}\\ \dot{\Psi}\end{bmatrix}$$

and the differential equation of the angle of rotation would be complete. Unfortunately, however, Ψ does not rotate about the z_f axis, but about the z_g axis and Θ does not rotate about the y_f axis, but about the k_2 node axis. The corresponding two angular derivatives must therefore first be transformed individually into the body-fixed axis system:

$$\begin{aligned}
\left(\frac{\mathrm{d}\boldsymbol{\Phi}}{\mathrm{d}t}\right)_f &= \begin{bmatrix}\dot{\Phi}\\ 0\\ 0\end{bmatrix}\\
&+ \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\Phi & \sin\Phi\\ 0 & -\sin\Phi & \cos\Phi\end{bmatrix}\begin{bmatrix}0\\ \dot{\Theta}\\ 0\end{bmatrix}\\
&+ \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\Phi & \sin\Phi\\ 0 & -\sin\Phi & \cos\Phi\end{bmatrix}\begin{bmatrix}\cos\Theta & 0 & -\sin\Theta\\ 0 & 1 & 0\\ \sin\Theta & 0 & \cos\Theta\end{bmatrix}\begin{bmatrix}0\\ 0\\ \dot{\Psi}\end{bmatrix}\\
&= \begin{bmatrix}1 & 0 & -\sin\Theta\\ 0 & \cos\Phi & \sin\Phi\cos\Theta\\ 0 & -\sin\Phi & \cos\Phi\cos\Theta\end{bmatrix}\begin{bmatrix}\dot{\Phi}\\ \dot{\Theta}\\ \dot{\Psi}\end{bmatrix}
\end{aligned}$$

The differential equation of the attitude vector is then:

$$\begin{bmatrix}1 & 0 & -\sin\Theta\\ 0 & \cos\Phi & \sin\Phi\cos\Theta\\ 0 & -\sin\Phi & \cos\Phi\cos\Theta\end{bmatrix}\begin{bmatrix}\dot{\Phi}\\ \dot{\Theta}\\ \dot{\Psi}\end{bmatrix} = \boldsymbol{\Omega_{Kf}} = \begin{bmatrix}p_{Kf}\\ q_{Kf}\\ r_{Kf}\end{bmatrix}$$

To solve for the derivative vector, the transformation matrix must be inverted. Unfortunately, it is not orthogonal (because the Euler angles rotate about axes that are not perpendicular to each other) and therefore cannot be inverted simply by transposing:

$$\dot{\boldsymbol{\Phi}} = \begin{bmatrix} \dot{\Phi} \\ \dot{\Theta} \\ \dot{\Psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\Phi\tan\Theta & \cos\Phi\tan\Theta \\ 0 & \cos\Phi & -\sin\Phi \\ 0 & {}^{\sin\Phi}\!/_{\cos\Theta} & {}^{\cos\Phi}\!/_{\cos\Theta} \end{bmatrix} \boldsymbol{\Omega_{Kf}} = \boldsymbol{M_{\Phi f}} \cdot \boldsymbol{\Omega_{Kf}} \tag{5.1}$$

5.5.3 Differential equation of the flight-path velocity vector

Momentum theorem: "The force $\boldsymbol{R}$ is the temporal change of the momentum $\boldsymbol{P}$":

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = \boldsymbol{R}$$

Expressed in the body-fixed axis system:

$$\left(\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t}\right)_f = \boldsymbol{R_f}$$

Problem: The body-fixed axis system in which the change in momentum

$$\left(\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t}\right)_f$$

is described, is not an inertial system, but rotates with the flight-path angular velocity $\boldsymbol{\Omega_K}$ with respect to the earth, which is assumed to be at rest. As shown in section 5.5.3.1, the rotation of the body-fixed axis system must therefore be taken into account in the so-called Euler term (cross product) for the inertial derivation of the momentum:

$$\dot{\boldsymbol{P}}_{\boldsymbol{f}} + \boldsymbol{\Omega_{Kf}} \times \boldsymbol{P_f} = \boldsymbol{R_f}$$

"Momentum is mass times velocity":

$$(m\boldsymbol{V_{Kf}})^{\cdot} + \boldsymbol{\Omega_{Kf}} \times (m\boldsymbol{V_{Kf}}) = \boldsymbol{R_f}$$

Product rule of differentiation:

$$\dot{m}\boldsymbol{V_{Kf}} + m\dot{\boldsymbol{V}}_{\boldsymbol{Kf}} + \boldsymbol{\Omega_{Kf}} \times (m\boldsymbol{V_{Kf}}) = \boldsymbol{R_f}$$

Neglecting the change in mass:

$$m\dot{\boldsymbol{V}}_{\boldsymbol{Kf}} + \boldsymbol{\Omega_{Kf}} \times (m\boldsymbol{V_{Kf}}) = \boldsymbol{R_f}$$

Factoring out the constant, scalar mass:

$$m\left(\dot{\boldsymbol{V}}_{\boldsymbol{Kf}}+\boldsymbol{\Omega}_{\boldsymbol{Kf}}\times\boldsymbol{V}_{\boldsymbol{Kf}}\right)=\boldsymbol{R}_{\boldsymbol{f}}$$

Total force from engine, aerodynamics and weight:

$$m\left(\dot{\boldsymbol{V}}_{\boldsymbol{Kf}}+\boldsymbol{\Omega}_{\boldsymbol{Kf}}\times\boldsymbol{V}_{\boldsymbol{Kf}}\right)=\boldsymbol{R}_{\boldsymbol{f}}^{\boldsymbol{F}}+\boldsymbol{R}_{\boldsymbol{f}}^{\boldsymbol{A}}+\boldsymbol{G}_{\boldsymbol{f}}$$

Transformation of weight and aerodynamic force:

$$m\left(\dot{\boldsymbol{V}}_{\boldsymbol{Kf}}+\boldsymbol{\Omega}_{\boldsymbol{Kf}}\times\boldsymbol{V}_{\boldsymbol{Kf}}\right)=\boldsymbol{R}_{\boldsymbol{f}}^{\boldsymbol{F}}+\boldsymbol{M}_{\boldsymbol{fa}}\boldsymbol{R}_{\boldsymbol{a}}^{\boldsymbol{A}}+\boldsymbol{M}_{\boldsymbol{fg}}\boldsymbol{G}_{\boldsymbol{g}}$$

Solve for the derivative:

$$\dot{\boldsymbol{V}}_{\boldsymbol{Kf}}=\frac{1}{m}\left(\boldsymbol{R}_{\boldsymbol{f}}^{\boldsymbol{F}}+\boldsymbol{M}_{\boldsymbol{fa}}\boldsymbol{R}_{\boldsymbol{a}}^{\boldsymbol{A}}+\boldsymbol{M}_{\boldsymbol{fg}}\boldsymbol{G}_{\boldsymbol{g}}\right)-\boldsymbol{\Omega}_{\boldsymbol{Kf}}\times\boldsymbol{V}_{\boldsymbol{Kf}}$$

“Canceling” out the mass from the weight:

$$\dot{\boldsymbol{V}}_{\boldsymbol{Kf}}=\frac{1}{m}\left(\boldsymbol{R}_{\boldsymbol{f}}^{\boldsymbol{F}}+\boldsymbol{M}_{\boldsymbol{fa}}\boldsymbol{R}_{\boldsymbol{a}}^{\boldsymbol{A}}\right)+\boldsymbol{M}_{\boldsymbol{fg}}\boldsymbol{g}_{\boldsymbol{g}}-\boldsymbol{\Omega}_{\boldsymbol{Kf}}\times\boldsymbol{V}_{\boldsymbol{Kf}}$$

5.5.3.1 Derivative of a vector in a rotating axis system

In the following, the geodetic axis system is understood as an inertial (stationary, space-fixed) axis system and the body-fixed axis system is used as an example of a non-inertial (rotating) axis system.

The inertial derivative (superscript g) of a vector $\boldsymbol{V}$ expressed in the body-fixed axis system (subscript f) is obtained by transforming the vector from the body-fixed axis system into the geodetic axis system, deriving it inertially there and then transforming it back into the aircraft axis system:

$$\left(\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}\right)_f^g=\boldsymbol{M}_{\boldsymbol{fg}}\frac{\mathrm{d}(\boldsymbol{M}_{\boldsymbol{gf}}\boldsymbol{V}_{\boldsymbol{f}})}{\mathrm{d}t}$$

Product rule:

$$\left(\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}\right)_f^g=\boldsymbol{M}_{\boldsymbol{fg}}\left(\boldsymbol{M}_{\boldsymbol{gf}}\left(\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}\right)_f^f+\frac{\mathrm{d}\left(\boldsymbol{M}_{\boldsymbol{gf}}\right)}{\mathrm{d}t}\boldsymbol{V}_{\boldsymbol{f}}\right)$$

Multiply out:

$$\left(\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}\right)_f^g=\boldsymbol{M}_{\boldsymbol{fg}}\boldsymbol{M}_{\boldsymbol{gf}}\left(\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}\right)_f^f+\boldsymbol{M}_{\boldsymbol{fg}}\frac{\mathrm{d}\left(\boldsymbol{M}_{\boldsymbol{gf}}\right)}{\mathrm{d}t}\boldsymbol{V}_{\boldsymbol{f}}$$

Combine matrices and use derivative points:

$$\left(\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}\right)_f^g = \dot{\boldsymbol{V}}_{\boldsymbol{f}} + \boldsymbol{M_{fg}}\dot{\boldsymbol{M}}_{\boldsymbol{gf}}\boldsymbol{V_f}$$

Thereby

$$\left(\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}\right)_f^f = \dot{\boldsymbol{V}}_{\boldsymbol{f}}$$

is the derivative of the vector performed component-wise directly in the body-fixed axis system, again expressed in the body-fixed axis system. The time derivative of the transformation matrix

$$\frac{\mathrm{d}\left(\boldsymbol{M_{gf}}\right)}{\mathrm{d}t} = \dot{\boldsymbol{M}}_{\boldsymbol{gf}}$$

is also performed individually for each element of the matrix.

In a somewhat longer derivation (Matlab file:
`http://buchholz.hs-bremen.de/rtfr/skript/euler_term.mlx`)
the matrix product containing the Euler angle derivatives

$$\boldsymbol{M_{fg}}\dot{\boldsymbol{M}}_{\boldsymbol{gf}}$$

can be combined to a cross product with the vector of the body-fixed flight-path angular velocity:

$$\left(\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}\right)_f^g = \dot{\boldsymbol{V}}_{\boldsymbol{f}} + \boldsymbol{\Omega_{Kf}} \times \boldsymbol{V_f}$$

The cross product $\boldsymbol{\Omega_{Kf}} \times \boldsymbol{V_f}$ is sometimes called the “Euler term”.

5.5.3.2 Example: horizontal (tethered) turn (without banking, without slipping)

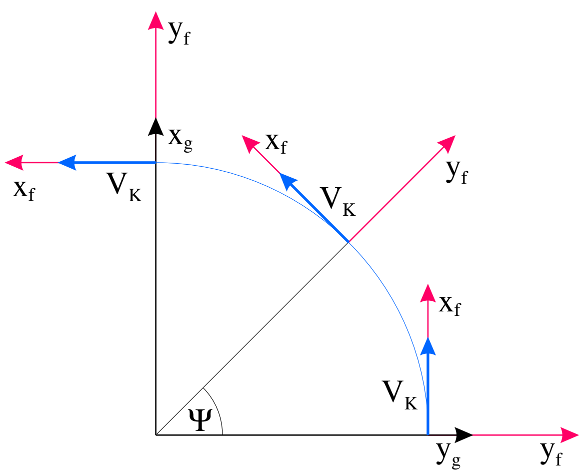


Figure 5.13: Rotation of the body-fixed axis system and the flight-path velocity vector during horizontal turn

Since the flight-path velocity vector $\boldsymbol{V_K}$ (tangential to the flight-path) rotates together with the aircraft (and thus also with the body-fixed axis system) in a horizontal, bank-free and slip-free turn, it always points in the x_f direction.

It is therefore constant when expressed in the body-fixed axis system:

$$\boldsymbol{V_{Kf}} = \begin{bmatrix} u_{Kf} \\ 0 \\ 0 \end{bmatrix}$$

Therefore, the derivative carried out component-wise directly in the body-fixed axis system vanishes:

$$\left(\frac{\mathrm{d}\boldsymbol{V_K}}{\mathrm{d}t}\right)_f^f = \dot{\boldsymbol{V}}_{\boldsymbol{Kf}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the flight-path lies in the earth's horizontal plane, the angular velocity vector can only have a z component which is equal to the temporal change of the yaw angle in the negative direction of rotation:

$$\boldsymbol{\Omega_{Kf}} = \begin{bmatrix} 0 \\ 0 \\ -\dot{\Psi} \end{bmatrix}$$

The complete inertial flight-path velocity vector derivative is then:

$$\left(\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}\right)_f^g = \dot{\boldsymbol{V}}_{\boldsymbol{f}} + \boldsymbol{\Omega_{Kf}} \times \boldsymbol{V_f} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\dot{\Psi} \end{bmatrix} \times \begin{bmatrix} u_{Kf} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\dot{\Psi} \cdot u_{Kf} \\ 0 \end{bmatrix}$$

The term $-\dot{\Psi} \cdot u_{Kf}$ corresponds, physically correct, exactly to the centripetal acceleration which acts in the negative y_f direction and keeps the aircraft on its circular path.

5.5.4 Differential equation of the flight-path angular velocity vector

The derivation of the angular speed differential equation is analogous to the derivation of the velocity differential equation. Only the forces have to be replaced by the moments, the momentum by the angular momentum, the velocity by the angular velocity and the scalar mass by the tensor of the moments of inertia:

Angular momentum theorem: "The moment $\boldsymbol{Q}$ is the temporal change of the angular momentum $\boldsymbol{D}$":

$$\frac{\mathrm{d}\boldsymbol{D}}{\mathrm{d}t} = \boldsymbol{Q}$$

Expressed in the body-fixed axis system:

$$\left(\frac{\mathrm{d}\boldsymbol{D}}{\mathrm{d}t}\right)_f = \boldsymbol{Q_f}$$

Inertial derivation:

$$\dot{\boldsymbol{D}}_{\boldsymbol{f}} + \boldsymbol{\Omega_{Kf}} \times \boldsymbol{D_f} = \boldsymbol{Q_f}$$

“Angular momentum is inertia tensor times angular velocity”:

$$\left(\boldsymbol{I_f} \cdot \boldsymbol{\Omega_{Kf}}\right)^{\cdot} + \boldsymbol{\Omega_{Kf}} \times \left(\boldsymbol{I_f} \cdot \boldsymbol{\Omega_{Kf}}\right) = \boldsymbol{Q_f}$$

Constant inertia tensor:

$$\boldsymbol{I_f} \cdot \dot{\boldsymbol{\Omega}}_{\boldsymbol{Kf}} + \boldsymbol{\Omega_{Kf}} \times \left(\boldsymbol{I_f} \cdot \boldsymbol{\Omega_{Kf}}\right) = \boldsymbol{Q_f}$$

Total moment from engines and aerodynamics

$$\boldsymbol{I_f} \cdot \dot{\boldsymbol{\Omega}}_{\boldsymbol{Kf}} + \boldsymbol{\Omega_{Kf}} \times \left(\boldsymbol{I_f} \cdot \boldsymbol{\Omega_{Kf}}\right) = \boldsymbol{Q_f^F} + \boldsymbol{Q_f^A}$$

Transformation of the aerodynamic moment vector:

$$\boldsymbol{I_f} \cdot \dot{\boldsymbol{\Omega}}_{\boldsymbol{Kf}} + \boldsymbol{\Omega_{Kf}} \times \left(\boldsymbol{I_f} \cdot \boldsymbol{\Omega_{Kf}}\right) = \boldsymbol{Q_f^F} + \boldsymbol{M_{fa}} \boldsymbol{Q_a^A}$$

Solve for the derivative:

$$\dot{\boldsymbol{\Omega}}_{\boldsymbol{Kf}} = \boldsymbol{I_f^{-1}} \cdot \left(\boldsymbol{Q_f^F} + \boldsymbol{M_{fa}} \boldsymbol{Q_a^A} - \boldsymbol{\Omega_{Kf}} \times \left(\boldsymbol{I_f} \cdot \boldsymbol{\Omega_{Kf}}\right)\right)$$

Unfortunately, the inertia tensor – as a matrix – cannot be excluded from the cross product. For the same reason, its reciprocal value is calculated by regular matrix inversion.

5.5.4.1 The inertia tensor

The inertia tensor describes, analogously to the mass in translational motions, the inertia with which the system resists a change in angular motion. However, while the mass as a scalar quantity is the same in all translational directions, the angular inertias differ depending on the axis of rotation considered.

In the body-fixed axis system, the (symmetric) inertia tensor is:

$$\boldsymbol{I_f} = \begin{bmatrix} I_{xf} & -I_{xyf} & -I_{xzf} \\ -I_{xyf} & I_{yf} & -I_{yzf} \\ -I_{xzf} & -I_{yzf} & I_{zf} \end{bmatrix}$$

On its main diagonal are the moments of inertia:

$$
\begin{aligned}
I_{xf} &= \int \left(y_f^2 + z_f^2\right) \cdot \mathrm{d}m \\
I_{yf} &= \int \left(x_f^2 + z_f^2\right) \cdot \mathrm{d}m \\
I_{zf} &= \int \left(x_f^2 + y_f^2\right) \cdot \mathrm{d}m
\end{aligned}
$$

In other words, a moment of inertia is the sum (integral) of all infinitesimally small mass particles multiplied by the square of their respective lever arm (Pythagorean distance from the corresponding axis of rotation).

The non-diagonal elements of the inertia tensor are called products of inertia:

$$
\begin{aligned}
I_{xyf} &= \int (x_f \cdot y_f) \cdot \mathrm{d}m \\
I_{xzf} &= \int (x_f \cdot z_f) \cdot \mathrm{d}m \\
I_{yzf} &= \int (y_f \cdot z_f) \cdot \mathrm{d}m
\end{aligned}
$$

In the case of a product of inertia, the mass particles are formally multiplied by two lever arms (position coordinates) each, which has the consequence that for symmetrical aircraft I_{xyf} and I_{yzf} disappear. If the x_f-z_f plane represents a plane of symmetry for an aircraft, for every mass particle on the right side of the plane of symmetry there is a corresponding (identical) mass particle on the left side. Both differ only by the sign of their y_f coordinate (right positive, left negative), so that the integral (the sum) always disappears if the integrand contains a y_f factor: Symmetrical aircraft:

$$
I_{xyf} = I_{yzf} = 0
$$

In addition to the moments of inertia, the inertia tensor then only contains the product of inertia I_{xzf}:

$$
\boldsymbol{I_f} = \begin{bmatrix} I_{xf} & 0 & -I_{xzf} \\ 0 & I_{yf} & 0 \\ -I_{xzf} & 0 & I_{zf} \end{bmatrix} \tag{5.2}
$$

If the body-fixed axes are defined in the direction of the main inertia axes, then (and only then) the last deviation moment also disappears:

$$
I_{xzf} = 0
$$

Effect of the deviation moments Transferring Newton's second axiom

$$
\boldsymbol{F} = m \cdot \boldsymbol{a}
$$

on the rotary motion, the force $\boldsymbol{F}$ becomes the moment $\boldsymbol{Q}$, the acceleration $\boldsymbol{a}$ becomes the angular acceleration $\dot{\boldsymbol{\Omega}}$ and the scalar mass m becomes the inertia tensor $\boldsymbol{I}$:

$$\boldsymbol{Q} = \boldsymbol{I} \cdot \dot{\boldsymbol{\Omega}}$$

Solving for the angular acceleration:

$$\dot{\boldsymbol{\Omega}} = \boldsymbol{I}^{-1}\boldsymbol{Q} \tag{5.3}$$

The inversion of the inertia tensor can still be performed relatively clearly in an analytical manner. The inverse of a matrix can be calculated from the quotient of its adjugate matrix and its determinant:

$$\boldsymbol{I}^{-1} = \frac{1}{|\boldsymbol{I}|} \cdot \boldsymbol{I_{adj}}$$

Using eq (5.2) the result is (with the index f omitted for the sake of clarity):

$$\begin{aligned}
\begin{bmatrix} I_x & 0 & -I_{xz} \\ 0 & I_y & 0 \\ -I_{xz} & 0 & I_z \end{bmatrix}^{-1} &= \frac{1}{I_x I_y I_z - I_{xz}^2 I_y} \begin{bmatrix} I_y I_z & 0 & I_{xz} I_y \\ 0 & I_x I_z - I_{xz}^2 & 0 \\ I_{xz} I_y & 0 & I_x I_y \end{bmatrix} \\
&= \begin{bmatrix} \frac{I_z}{I_x I_z - I_{xz}^2} & 0 & \frac{I_{xz}}{I_x I_z - I_{xz}^2} \\ 0 & \frac{1}{I_y} & 0 \\ \frac{I_{xz}}{I_x I_z - I_{xz}^2} & 0 & \frac{I_x}{I_x I_z - I_{xz}^2} \end{bmatrix}
\end{aligned}$$

Now, if this inverse inertia tensor is used to establish the relationship between angular acceleration $\dot{\boldsymbol{\Omega}}$ and angular moment $\boldsymbol{Q}$ in eq (5.3)

$$\begin{aligned}
\dot{\boldsymbol{\Omega}} &= \boldsymbol{I}^{-1} \cdot \boldsymbol{Q} \\
\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} &= \begin{bmatrix} \frac{I_z}{I_x I_z - I_{xz}^2} & 0 & \frac{I_{xz}}{I_x I_z - I_{xz}^2} \\ 0 & \frac{1}{I_y} & 0 \\ \frac{I_{xz}}{I_x I_z - I_{xz}^2} & 0 & \frac{I_x}{I_x I_z - I_{xz}^2} \end{bmatrix} \begin{bmatrix} L \\ M \\ N \end{bmatrix}
\end{aligned}$$

the first line shows

$$\dot{p} = \frac{I_z}{I_x I_z - I_{xz}^2} L + \frac{I_{xz}}{I_x I_z - I_{xz}^2} N$$

that a roll acceleration $\dot{p}$ is not only generated by a roll moment L, but that a yaw moment N also contributes to the total roll acceleration via the deviation moment I_{xz}. Argued the other way round, a pure yaw moment not only causes a yaw acceleration but also a parasitic roll acceleration.

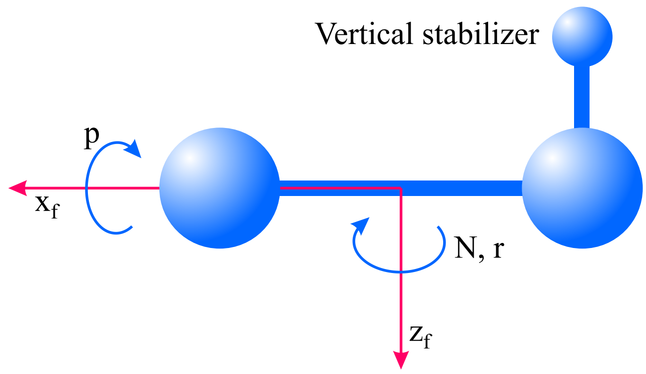


Figure 5.14: With a pure yaw moment, the aircraft also reacts with a roll acceleration (in addition to the yaw acceleration) because of the "inertia" of the vertical stabilizer.

5.6 Quaternions

The non-orthogonal transformation matrix in the differential equation of the attitude vector (5.1) contains the quotients

$$\sin\Phi\tan\Theta = \frac{\sin\Phi\sin\Theta}{\cos\Theta}, \quad \cos\Phi\tan\Theta = \frac{\cos\Phi\sin\Theta}{\cos\Theta}, \quad \frac{\sin\Phi}{\cos\Theta}, \quad \frac{\cos\Phi}{\cos\Theta}$$

whose denominator disappears for a pitch angle Θ of $\pm\frac{\pi}{2}$. If additionally the roll angle Φ is a multiple of $\frac{\pi}{2}$, the corresponding quotient is indefinite $\left(\frac{0}{0}\right)$, otherwise the quotients take on infinitely large values. A simulation terminates in all cases with an error message.

From a physical perspective, the indeterminacy problem manifests itself, for example, in an airplane oriented vertically downwards $\left(\Theta = -\frac{\pi}{2}\right)$, through the coincidence of the geodetic z axis and the body-fixed x axis.

Both a "yaw" with $\dot{\Psi}$ about the z_g axis as well as a "roll" with $\dot{\Phi}$ about the x_f axis now lead to the same motion about the vertical aircraft longitudinal axis (gimbal lock).

To solve the problem, one can use the four components of a quaternion $[\,a\ b\ c\ d\,]$ as state variables in the differential equation of the attitude vector instead of the three Euler angles $[\,\Phi\ \Theta\ \Psi\,]$.

5.6.1 Properties of the quaternions

Quaternions are – similar to the complex numbers – an extension of the real numbers. While a complex number z consists of a real part a and a **scalar** imaginary part b ($z = a + b \cdot \mathrm{i}$), a quaternion Z has a real part a and a **vectorial** imaginary part $[\,b\ c\ d\,]$, whose components are each multiplied by their own imaginary unit i, j and k:

$$Z = a + b \cdot \mathrm{i} + c \cdot \mathrm{j} + d \cdot \mathrm{k}$$

The imaginary units are defined as for the complex numbers:

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1$$

In addition, the product of two different imaginary units results in the third and it is anti-commutative (change of sign when the order is reversed):

$$
\begin{array}{lll}
\mathrm{i} \cdot \mathrm{j} = \mathrm{k} & \mathrm{j} \cdot \mathrm{k} = \mathrm{i} & \mathrm{k} \cdot \mathrm{i} = \mathrm{j} \\
\mathrm{j} \cdot \mathrm{i} = -\mathrm{k} & \mathrm{k} \cdot \mathrm{j} = -\mathrm{i} & \mathrm{i} \cdot \mathrm{k} = -\mathrm{j}
\end{array}
$$

The sum of two quaternions is calculated component-wise

$$
\begin{aligned}
Z_1 + Z_2 &= (a_1 + b_1\mathrm{i} + c_1\mathrm{j} + d_1\mathrm{k}) + (a_2 + b_2\mathrm{i} + c_2\mathrm{j} + d_2\mathrm{k}) \\
&= (a_1 + a_2) + (b_1 + b_2)\,\mathrm{i} + (c_1 + c_2)\,\mathrm{j} + (d_1 + d_2)\,\mathrm{k}
\end{aligned}
$$

whereas with the quaternion product the signs of the products of the imaginary units must be taken into account:

$$
\begin{aligned}
Z_1 \cdot Z_2 &= (a_1 + b_1\mathrm{i} + c_1\mathrm{j} + d_1\mathrm{k}) \cdot (a_2 + b_2\mathrm{i} + c_2\mathrm{j} + d_2\mathrm{k}) \\
&= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) \\
&\quad + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)\,\mathrm{i} \\
&\quad + (a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)\,\mathrm{j} \\
&\quad + (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)\,\mathrm{k}
\end{aligned}
$$

The conjugate quaternion $\overline{Z}$ results – as with the complex numbers – from a negative sign in the imaginary part:

$$
\overline{Z} = \overline{a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}} = a - (b\mathrm{i} + c\mathrm{j} + d\mathrm{k}) = a - b\mathrm{i} - c\mathrm{j} - d\mathrm{k} \tag{5.4}
$$

The product of a quaternion with its conjugate is purely real

$$
\begin{aligned}
Z \cdot \overline{Z} &= (a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}) \cdot (a - b\mathrm{i} - c\mathrm{j} - d\mathrm{k}) \\
&= (aa + bb + cc + dd) \\
&\quad + (-ab + ba - cd + dc)\,\mathrm{i} \\
&\quad + (-ac + bd + ca - db)\,\mathrm{j} \\
&\quad + (-ad - bc + cb + da)\,\mathrm{k} \\
&= a^2 + b^2 + c^2 + d^2
\end{aligned}
$$

and corresponds to the norm squared (magnitude squared) $|Z|^2$ of the quaternion:

$$
|Z| = \sqrt{Z \cdot \overline{Z}} = \sqrt{a^2 + b^2 + c^2 + d^2}
$$

Any quaternion with a non-zero length can be transformed into its unit quaternion by dividing it by its magnitude:

$$
Z^0 = \frac{Z}{|Z|} = \frac{a}{|Z|} + \frac{b}{|Z|}\mathrm{i} + \frac{c}{|Z|}\mathrm{j} + \frac{d}{|Z|}\mathrm{k} \tag{5.5}
$$

5.6.2 Calculation of the quaternion from the angle of rotation and the axis of rotation

In eq (4.1) the transformation matrix $\boldsymbol{M_{fg}}$ is defined, which consists of trigonometric functions of the Euler angles Φ, Θ, and Ψ, and which transforms a vector $\boldsymbol{v} = [\,x\ y\ z\,]^T$ from the geodetic (index g) to the body-fixed (index f) axis system:

$$\begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} = \begin{bmatrix} \cos\Theta\cos\Psi & \cos\Theta\sin\Psi & -\sin\Theta \\ \sin\Phi\sin\Theta\cos\Psi - \cos\Phi\sin\Psi & \sin\Phi\sin\Theta\sin\Psi + \cos\Phi\cos\Psi & \sin\Phi\cos\Theta \\ \cos\Phi\sin\Theta\cos\Psi + \sin\Phi\sin\Psi & \cos\Phi\sin\Theta\sin\Psi - \sin\Phi\cos\Psi & \cos\Phi\cos\Theta \end{bmatrix} \begin{bmatrix} x_g \\ y_g \\ z_g \end{bmatrix} \tag{5.6}$$

The same transformation can also be realised with a quaternion.

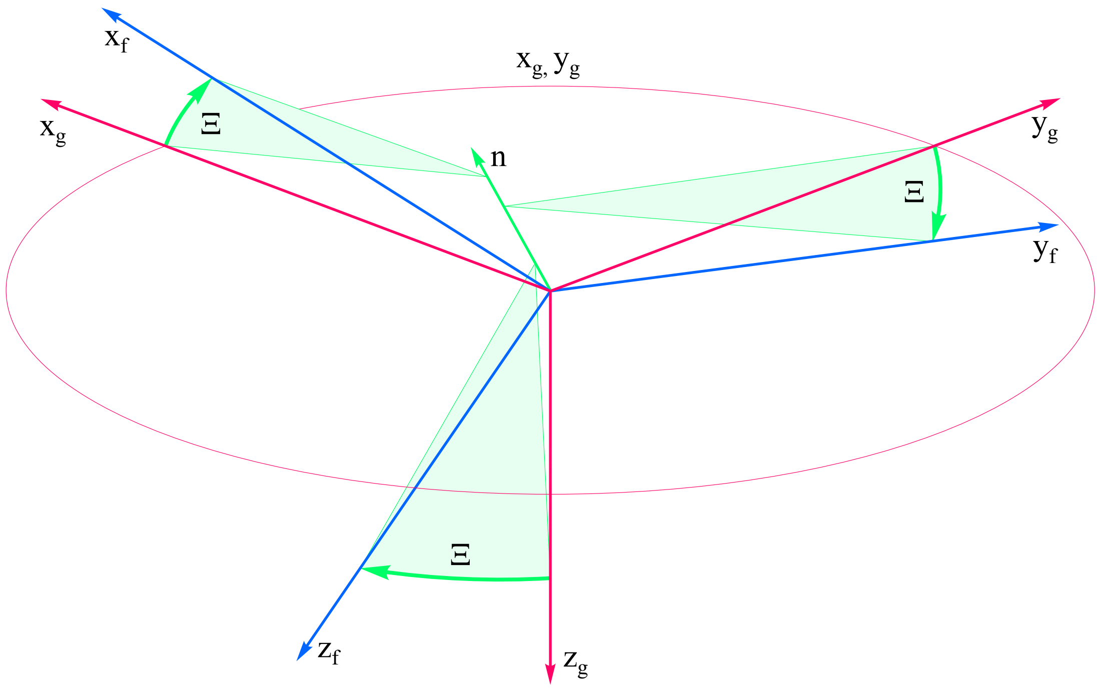


Figure 5.15: Total rotation from the geodetic to the body-fixed axis system with the angle Ξ about the rotation axis $\boldsymbol{n}$

For this purpose, the three individual rotations with Ψ, Θ, and Φ shown in figure 4.8, which are necessary to transfer the geodetic axis system into the body-fixed axis system, are combined into the total rotation with the angle Ξ about the rotation axis $\boldsymbol{n} = [\,n_x\ n_y\ n_z\,]^T$ shown in figure 5.15. (The rotation axis has the same coordinates in both axis systems, because the body-fixed axis system rotates exactly about the rotation axis and the coordinates of the rotation axis do not change).

Provided that the rotation axis vector $\boldsymbol{n}$ is a unit vector (i. e. its length has been normalised to one according to eq (5.5)), the corresponding unit quaternion Z_D can be calculated from the angle of rotation and axis of rotation:

$$
\begin{aligned}
Z_D &= a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} \\
&= \cos\left(\frac{\Xi}{2}\right) + n_x \cdot \sin\left(\frac{\Xi}{2}\right)\mathrm{i} + n_y \cdot \sin\left(\frac{\Xi}{2}\right)\mathrm{j} + n_z \cdot \sin\left(\frac{\Xi}{2}\right)\mathrm{k}
\end{aligned} \tag{5.7}
$$

5.6.3 Calculation of the angle of rotation and the axis of rotation from the quaternion

For a given quaternion $Z_D = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$, the corresponding rotation angle Ξ and the rotation axis $\boldsymbol{n} = [\, n_x \; n_y \; n_z \,]^T$ are calculated directly from eq (5.7). The real part of the quaternion provides the angle of rotation

$$
a = \cos\left(\frac{\Xi}{2}\right) \quad \Rightarrow \quad \Xi = 2 \arccos(a)
$$

With the angle of rotation just calculated, the axis of rotation then results from the imaginary part of the quaternion:

$$
\begin{aligned}
b\mathrm{i} + c\mathrm{j} + d\mathrm{k} &= n_x \cdot \sin\left(\frac{\Xi}{2}\right)\mathrm{i} + n_y \cdot \sin\left(\frac{\Xi}{2}\right)\mathrm{j} + n_z \cdot \sin\left(\frac{\Xi}{2}\right)\mathrm{k} \\
\Rightarrow \quad \boldsymbol{n} &= \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} \dfrac{b}{\sin\left(\frac{\Xi}{2}\right)} \\ \dfrac{c}{\sin\left(\frac{\Xi}{2}\right)} \\ \dfrac{d}{\sin\left(\frac{\Xi}{2}\right)} \end{bmatrix}
\end{aligned}
$$

5.6.4 Calculation of the Euler angles from the transformation matrix

From individual elements of the transformation matrix in eq (5.6), equations are obtained to calculate the Euler angles:

$$M_{13} = -\sin\Theta \tag{5.8}$$
$$M_{11} = \cos\Theta \cos\Psi \tag{5.9}$$
$$M_{12} = \cos\Theta \sin\Psi \tag{5.10}$$
$$M_{23} = \sin\Phi \cos\Theta \tag{5.11}$$
$$M_{33} = \cos\Phi \cos\Theta \tag{5.12}$$

The pitch angle follows directly from eq (5.8):

$$-\sin\Theta = M_{13} \quad \Rightarrow \quad \Theta = -\arcsin M_{13} \tag{5.13}$$

From the quotient of eqs (5.10) and (5.9) the yaw angle is calculated

$$\frac{\cos\Theta\sin\Psi}{\cos\Theta\cos\Psi} = \tan\Psi = \frac{M_{12}}{M_{11}} \quad \Rightarrow \quad \Psi = \arctan\left(\frac{M_{12}}{M_{11}}\right) \tag{5.14}$$

and the quotient of eqs (5.11) and (5.12) leads to the bank angle:

$$\frac{\sin\Phi\cos\Theta}{\cos\Phi\cos\Theta} = \tan\Phi = \frac{M_{23}}{M_{33}} \quad \Rightarrow \quad \Phi = \arctan\left(\frac{M_{23}}{M_{33}}\right) \tag{5.15}$$

In order to obtain the full range of angles $(-\pi\ldots\pi)$ for Ψ and Φ, the `atan2` function – available in most programming languages – must be used, which can also handle the singularities that occur with a "normal" arc tangent when the denominators of eqs (5.14) or (5.15) disappear because an angle is $\frac{\pi}{2}$. In some publications it is suggested that after calculating the pitch angle according to eq (5.13), it should be inserted into eqs (5.9) - (5.12) in order to calculate Ψ and Φ with arcsin or arccos functions. In this way, however, Ψ and Φ would be incorrectly restricted to the ranges $(-\frac{\pi}{2}\ldots\frac{\pi}{2})$ and $(0\ldots\pi)$ respectively, since arcsin and arccos are only able to deliver values in these ranges.

5.6.5 Calculation of the transformation matrix from the quaternion

The conjugate quaternion $\overline{Z_D}$ to the quaternion defined in eq (5.7) is given by eq (5.4)

$$\overline{Z_D} = a - b\mathrm{i} - c\mathrm{j} - d\mathrm{k}$$

Also, the vector to be transformed $\boldsymbol{v_g} = [\,x_g\ y_g\ z_g\,]^T$ is given in the form of a quaternion Z_g:

$$Z_g = 0 + x_g\mathrm{i} + y_g\mathrm{j} + z_g\mathrm{k}$$

The transformation to the body-fixed axis system analogous to eq (5.6) is then carried out by means of two quaternion products

$$\begin{aligned}
Z_f &= \overline{Z_D}\cdot Z_g\cdot Z_D\\
&= (a - b\mathrm{i} - c\mathrm{j} - d\mathrm{k})\cdot(0 + x_g\mathrm{i} + y_g\mathrm{j} + z_g\mathrm{k})\cdot(a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k})\\
&= 0\\
&\quad + \Big(\Big(a^2 + b^2 - c^2 - d^2\Big)\,x_g + 2\,(bc + ad)\,y_g + 2\,(bd - ac)\,z_g\Big)\,\mathrm{i}\\
&\quad + \Big(2\,(bc - ad)\,x_g + \Big(a^2 - b^2 + c^2 - d^2\Big)\,y_g + 2\,(cd + ab)\,z_g\Big)\,\mathrm{j}\\
&\quad + \Big(2\,(bd + ac)\,x_g + 2\,(cd - ab)\,y_g + \Big(a^2 - b^2 - c^2 + d^2\Big)\,z_g\Big)\,\mathrm{k}
\end{aligned} \tag{5.16}$$

whereby the quaternion

$$Z_f = 0 + x_f \mathrm{i} + y_f \mathrm{j} + z_f \mathrm{k}$$

contains the components of the transformed vector $\boldsymbol{v_f} = [\,x_f \; y_f \; z_f\,]^T$. Eq (5.16) can be presented more clearly in matrix notation

$$\begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} = \begin{bmatrix} (a^2+b^2-c^2-d^2) & 2\,(bc+ad) & 2\,(bd-ac) \\ 2\,(bc-ad) & (a^2-b^2+c^2-d^2) & 2\,(cd+ab) \\ 2\,(bd+ac) & 2\,(cd-ab) & (a^2-b^2-c^2+d^2) \end{bmatrix} \begin{bmatrix} x_g \\ y_g \\ z_g \end{bmatrix} \tag{5.17}$$

so that the transformation matrix can be calculated directly from the quaternion components:

$$\boldsymbol{M_{fg}} = \begin{bmatrix} (a^2+b^2-c^2-d^2) & 2\,(bc+ad) & 2\,(bd-ac) \\ 2\,(bc-ad) & (a^2-b^2+c^2-d^2) & 2\,(cd+ab) \\ 2\,(bd+ac) & 2\,(cd-ab) & (a^2-b^2-c^2+d^2) \end{bmatrix} \tag{5.18}$$

If the Euler angles are not explicitly required in a simulation, the transformation matrix $\boldsymbol{M_{fg}}$ built up from the components of the quaternion from eq (5.18) can be used directly in the differential equations of the flight-path velocity and the position, so that no trigonometric functions of the Euler angles have to be calculated and the computational effort is reduced.

5.6.6 Calculation of the Euler angles from the quaternion

By inserting the corresponding elements of the transformation matrix from eq (5.18) into eqs (5.13) - (5.15), the Euler angles can be calculated directly from the quaternion components:

$$\Theta = -\arcsin M_{13} = -\arcsin\left(2\,(bd-ac)\right) = \arcsin\left(2\,(ac-bd)\right) \tag{5.19}$$

$$\Psi = \arctan\left(\frac{M_{12}}{M_{11}}\right) = \arctan\left(\frac{2\,(bc+ad)}{a^2+b^2-c^2-d^2}\right) \tag{5.20}$$

$$\Phi = \arctan\left(\frac{M_{23}}{M_{33}}\right) = \arctan\left(\frac{2\,(cd+ab)}{a^2-b^2-c^2+d^2}\right) \tag{5.21}$$

Of course, in a numerical implementation of eqs (5.19) - (5.21), the `atan2` function must also be used here.

5.6.7 Calculation of the quaternion from the Euler angles

The transformation from the geodetic to the body-fixed axis system shown in eq (5.16) with the aid of the total quaternion Z_D can also be built up from the inside outwards from the three individual quaternions Z_Ψ , Z_Θ and Z_Φ:

$$
\begin{aligned}
Z_f &= \overline{Z_\Phi} \cdot \left(\overline{Z_\Theta} \cdot \left(\overline{Z_\Psi} \cdot Z_g \cdot Z_\Psi\right) \cdot Z_\Theta\right) \cdot Z_\Phi \\
&= \left(\overline{Z_\Phi} \cdot \overline{Z_\Theta} \cdot \overline{Z_\Psi}\right) \cdot Z_g \cdot (Z_\Psi \cdot Z_\Theta \cdot Z_\Phi) \\
&= \underbrace{\overline{(Z_\Psi \cdot Z_\Theta \cdot Z_\Phi)}}_{\overline{Z_D}} \cdot Z_g \cdot \underbrace{(Z_\Psi \cdot Z_\Theta \cdot Z_\Phi)}_{Z_D}
\end{aligned} \tag{5.22}
$$

The associative law that applies to quaternions is taken into account as well as the fact that the conjugate of a quaternion product is equal to the product of the individual conjugates in reverse order.

The single quaternions are constructed according to eq (5.7) from the respective rotation angles (Ψ, Θ, and Φ) and the corresponding rotation axes ($\boldsymbol{n_\Psi} = [\,0\ 0\ 1\,]^T, \ldots$):

$$
\begin{aligned}
Z_\Psi &= \cos\left(\frac{\Psi}{2}\right) + 0 \cdot \sin\left(\frac{\Psi}{2}\right)\mathrm{i} + 0 \cdot \sin\left(\frac{\Psi}{2}\right)\mathrm{j} + 1 \cdot \sin\left(\frac{\Psi}{2}\right)\mathrm{k} \\
&= \cos\left(\frac{\Psi}{2}\right) + \sin\left(\frac{\Psi}{2}\right)\mathrm{k} \\
Z_\Theta &= \cos\left(\frac{\Theta}{2}\right) + 0 \cdot \sin\left(\frac{\Theta}{2}\right)\mathrm{i} + 1 \cdot \sin\left(\frac{\Theta}{2}\right)\mathrm{j} + 0 \cdot \sin\left(\frac{\Theta}{2}\right)\mathrm{k} \\
&= \cos\left(\frac{\Theta}{2}\right) + \sin\left(\frac{\Theta}{2}\right)\mathrm{j} \\
Z_\Phi &= \cos\left(\frac{\Phi}{2}\right) + 1 \cdot \sin\left(\frac{\Phi}{2}\right)\mathrm{i} + 0 \cdot \sin\left(\frac{\Phi}{2}\right)\mathrm{j} + 0 \cdot \sin\left(\frac{\Phi}{2}\right)\mathrm{k} \\
&= \cos\left(\frac{\Phi}{2}\right) + \sin\left(\frac{\Phi}{2}\right)\mathrm{i}
\end{aligned}
$$

According to eq (5.22), the total quaternion Z_D is then given as a function of the Euler angles:

$$
\begin{aligned}
Z_D &= Z_\Psi \cdot Z_\Theta \cdot Z_\Phi \\
&= \left(\cos\left(\frac{\Psi}{2}\right) + \sin\left(\frac{\Psi}{2}\right)\mathrm{k}\right)\left(\cos\left(\frac{\Theta}{2}\right) + \sin\left(\frac{\Theta}{2}\right)\mathrm{j}\right)\left(\cos\left(\frac{\Phi}{2}\right) + \sin\left(\frac{\Phi}{2}\right)\mathrm{i}\right) \\
&= \cos\left(\frac{\Psi}{2}\right)\cos\left(\frac{\Theta}{2}\right)\cos\left(\frac{\Phi}{2}\right) + \sin\left(\frac{\Psi}{2}\right)\sin\left(\frac{\Theta}{2}\right)\sin\left(\frac{\Phi}{2}\right) \\
&\quad + \left(\cos\left(\frac{\Psi}{2}\right)\cos\left(\frac{\Theta}{2}\right)\sin\left(\frac{\Phi}{2}\right) - \sin\left(\frac{\Psi}{2}\right)\sin\left(\frac{\Theta}{2}\right)\cos\left(\frac{\Phi}{2}\right)\right)\mathrm{i} \\
&\quad + \left(\cos\left(\frac{\Psi}{2}\right)\sin\left(\frac{\Theta}{2}\right)\cos\left(\frac{\Phi}{2}\right) + \sin\left(\frac{\Psi}{2}\right)\cos\left(\frac{\Theta}{2}\right)\sin\left(\frac{\Phi}{2}\right)\right)\mathrm{j} \\
&\quad + \left(\sin\left(\frac{\Psi}{2}\right)\cos\left(\frac{\Theta}{2}\right)\cos\left(\frac{\Phi}{2}\right) - \cos\left(\frac{\Psi}{2}\right)\sin\left(\frac{\Theta}{2}\right)\sin\left(\frac{\Phi}{2}\right)\right)\mathrm{k}
\end{aligned} \tag{5.23}
$$

5.6.8 Calculation of the quaternion from the transformation matrix

The pragmatic way to calculate the quaternion from the transformation matrix with the tools presented so far would first determine the Euler angles from the transformation matrix and then the quaternion from the Euler angles. However, since the known problems with the Euler angles occur (computationally time-intensive trigonometric functions, "gimbal lock", cf. section 5.6), a direct alternative is explained below.

Eq (5.18) represents the transformation matrix $\boldsymbol{M_{fg}}$ as a function of the quaternion components a, b, c, and d. To calculate the first quaternion component a, the main diagonal elements of the transformation matrix are summed up:

$$\begin{aligned} M_{11} + M_{22} + M_{33} &= \left(a^2 + b^2 - c^2 - d^2\right) + \left(a^2 - b^2 + c^2 - d^2\right) + \left(a^2 - b^2 - c^2 + d^2\right) \\ &= 3a^2 - b^2 - c^2 - d^2 \end{aligned} \tag{5.24}$$

The condition that the quaternion has a magnitude of one

$$a^2 + b^2 + c^2 + d^2 = 1$$

can be solved for b^2

$$b^2 = 1 - a^2 - c^2 - d^2$$

and substituted in eq (5.24)

$$M_{11} + M_{22} + M_{33} = 3a^2 - \left(1 - a^2 - c^2 - d^2\right) - c^2 - d^2 = 4a^2 - 1 \tag{5.25}$$

so that eq (5.25) can be solved for the sought quaternion component

$$a = \frac{\sqrt{M_{11} + M_{22} + M_{33} + 1}}{2} \tag{5.26}$$

Cleverly chosen differences of two matrix elements

$$M_{23} - M_{32} = 2\,(cd + ab) - 2\,(cd - ab) = 4ab$$

each provide a determining equation for the remaining quaternion components:

$$b = \frac{M_{23} - M_{32}}{4a} \qquad c = \frac{M_{31} - M_{13}}{4a} \qquad d = \frac{M_{12} - M_{21}}{4a} \tag{5.27}$$

Unfortunately, when calculating the quaternion according to eqs (5.26) - (5.27), there is the problem that the quaternion component a – which is present in eq (5.27) in all denominators – can become zero, so that the other components can no longer be calculated. The component a, which – according to eq (5.23) – results from

$$a = \cos\left(\frac{\Psi}{2}\right)\cos\left(\frac{\Theta}{2}\right)\cos\left(\frac{\Phi}{2}\right) + \sin\left(\frac{\Psi}{2}\right)\sin\left(\frac{\Theta}{2}\right)\sin\left(\frac{\Phi}{2}\right)$$

disappears, for example, if one Euler angle has the value 0 and another the value π, since then both a cosine and a sine and thus both summands become zero. In this case, the calculation can begin with the second quaternion component b by summing the diagonal elements of the transformation matrix with different signs

$$\begin{aligned} M_{11} - M_{22} - M_{33} &= \left(a^2 + b^2 - c^2 - d^2\right) - \left(a^2 - b^2 + c^2 - d^2\right) - \left(a^2 - b^2 - c^2 + d^2\right) \\ &= -a^2 + 3b^2 - c^2 - d^2 \end{aligned}$$

so that, using

$$a^2 = 1 - b^2 - c^2 - d^2$$

yields a determining equation for b:

$$\begin{aligned} M_{11} - M_{22} - M_{33} &= -\left(1 - b^2 - c^2 - d^2\right) + 3b^2 - c^2 - d^2 = 4b^2 - 1 \\ \Rightarrow \quad b &= \frac{\sqrt{M_{11} - M_{22} - M_{33} + 1}}{2} \end{aligned}$$

With b, the other components can then also be determined

$$c = \frac{M_{12} + M_{21}}{4b} \qquad d = \frac{M_{13} + M_{31}}{4b} \qquad a = \frac{M_{23} - M_{32}}{4b}$$

In total, there are four sets of determining equations, depending on which diagonal element signs and thus which quaternion component is started with:

$$\begin{array}{llll} a = \dfrac{\sqrt{M_{11} + M_{22} + M_{33} + 1}}{2} & b = \dfrac{M_{23} - M_{32}}{4a} & c = \dfrac{M_{31} - M_{13}}{4a} & d = \dfrac{M_{12} - M_{21}}{4a} \\ b = \dfrac{\sqrt{M_{11} - M_{22} - M_{33} + 1}}{2} & c = \dfrac{M_{12} + M_{21}}{4b} & d = \dfrac{M_{13} + M_{31}}{4b} & a = \dfrac{M_{23} - M_{32}}{4b} \\ c = \dfrac{\sqrt{-M_{11} + M_{22} - M_{33} + 1}}{2} & d = \dfrac{M_{23} + M_{32}}{4c} & a = \dfrac{M_{31} - M_{13}}{4c} & b = \dfrac{M_{12} + M_{21}}{4c} \\ d = \dfrac{\sqrt{-M_{11} - M_{22} + M_{33} + 1}}{2} & a = \dfrac{M_{12} - M_{21}}{4d} & b = \dfrac{M_{13} + M_{31}}{4d} & c = \dfrac{M_{23} + M_{32}}{4d} \end{array}$$

Since the quaternion to be calculated is a unit quaternion, at least one of its components must be significantly different from zero so that it can be started with. In numerical practice, one can simply search for the largest radicant $\pm M_{11} \pm M_{22} \pm M_{33} + 1$ to decide which set of governing equations to use.

5.6.9 Differential equation of the quaternions

Under the important assumption that the quaternion $Z = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$ is a unit quaternion, the attitude differential eq (5.1) can be replaced by the very compact quaternion differential equation:

$$\dot{Z} = \frac{1}{2} \cdot Z \cdot Z_{\Omega} \tag{5.28}$$

Here Z_Ω is a pure quaternion (with vanishing real part) whose imaginary part consists of the three elements p_{Kf}, q_{Kf}, and r_{Kf} of the body-fixed angular velocity vector $\boldsymbol{\Omega_{Kf}}$:

$$Z_{\Omega} = 0 + p_{Kf}\mathrm{i} + q_{Kf}\mathrm{j} + r_{Kf}\mathrm{k}$$

On the right-hand side of the quaternion differential eq (5.28), the quaternion product can be multiplied out:

$$\begin{aligned}
\dot{Z} &= \frac{1}{2} \cdot Z \cdot Z_{\Omega} \\
&= \frac{1}{2} \cdot (a + b\mathrm{i} + c\mathrm{j} + dk) \cdot (0 + p_{Kf}\mathrm{i} + q_{Kf}\mathrm{j} + r_{Kf}\mathrm{k}) \\
&= \frac{1}{2} \cdot \{\, (-p_{Kf} \cdot b - q_{Kf} \cdot c - r_{Kf} \cdot d) \\
&\quad + (p_{Kf} \cdot a + r_{Kf} \cdot c - q_{Kf} \cdot d)\,\mathrm{i} \\
&\quad + (q_{Kf} \cdot a - r_{Kf} \cdot b + p_{Kf} \cdot d)\,\mathrm{j} \\
&\quad + (r_{Kf} \cdot a + q_{Kf} \cdot b - p_{Kf} \cdot c)\,\mathrm{k}\, \}
\end{aligned} \tag{5.29}$$

5.6.10 Numerical simulation

Often – for example for numerical simulation – eq (5.29) is represented in matrix notation by expressing the quaternion $Z = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$ in terms of a real column vector $\boldsymbol{Z} = [\, a\ b\ c\ d\,]^T$ and the right-hand side of eq (5.29) as a matrix-vector product:

$$\dot{\boldsymbol{Z}} = \begin{bmatrix} \dot{a} \\ \dot{b} \\ \dot{c} \\ \dot{d} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -p_{Kf} & -q_{Kf} & -r_{Kf} \\ p_{Kf} & 0 & r_{Kf} & -q_{Kf} \\ q_{Kf} & -r_{Kf} & 0 & p_{Kf} \\ r_{Kf} & q_{Kf} & -p_{Kf} & 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \frac{1}{2} \cdot \boldsymbol{M_{\Omega}} \cdot \boldsymbol{Z} \tag{5.30}$$

During a longer simulation, unavoidable numerical errors can lead to the prerequisite for eq (5.30), namely the fact that Z is a unit quaternion, no longer being fulfilled. In order to keep this error as small as possible, a simple proportional control is suitable. The control error $\Delta Z = 1 - |\boldsymbol{Z}|$, i. e. the deviation of the quaternion's magnitude from one, is thereby scaled with the current state $\boldsymbol{Z}$ and fed back to all components of the quaternion integrator via a proportional controller with the gain K. The extended eq (5.30) then reads:

$$
\begin{aligned}
\dot{\boldsymbol{Z}} &= \frac{1}{2} \cdot \boldsymbol{M_\Omega} \cdot \boldsymbol{Z} + K \cdot \Delta Z \cdot \boldsymbol{Z} \\
&= \frac{1}{2} \cdot \boldsymbol{M_\Omega} \cdot \boldsymbol{Z} + K \left(1 - |\boldsymbol{Z}|\right) \boldsymbol{Z} \\
&= \frac{1}{2} \cdot \boldsymbol{M_\Omega} \cdot \boldsymbol{Z} + K \left(1 - \sqrt{a^2 + b^2 + c^2 + d^2}\right) \boldsymbol{Z}
\end{aligned} \tag{5.31}
$$

To save computing time, the magnitude in eq (5.31) can be replaced by the magnitude squared without qualitatively changing the control behaviour:

$$
\dot{\boldsymbol{Z}} = \frac{1}{2} \cdot \boldsymbol{M_\Omega} \cdot \boldsymbol{Z} + K \left(1 - \left(a^2 + b^2 + c^2 + d^2\right)\right) \boldsymbol{Z} \tag{5.32}
$$

As with any controller design, a compromise has to be found for the controller gain K. If K is too small, the quaternion amount can deviate significantly from one, while a too large K leads to a stiffening of the system to be simulated, with the consequence of longer computation times and the danger of numerical instability.

The presented method for preserving the quaternion unit length has the advantage that it is not necessary to directly change the quaternion – which is part of the state vector of the simulation. However, if the simulation environment allows the direct setting of state variables without major effort, the quaternion can simply be made into a unit quaternion again after each integration step by dividing it by its magnitude.

Chapter 6

Controller design

6.1 Natural response

6.1.1 Division of the state variables into longitudinal and lateral motion

	$\boldsymbol{\Omega_K}$	$\boldsymbol{V_K}$	$\boldsymbol{\Phi}$	$\boldsymbol{s}$
Longitudinal motion	q_K	u_K w_K	Θ	x z
Lateral motion	p_K r_K	v_K	Φ Ψ	y

Table 6.1: State variables of longitudinal and lateral motion

6.1.2 Longitudinal motion

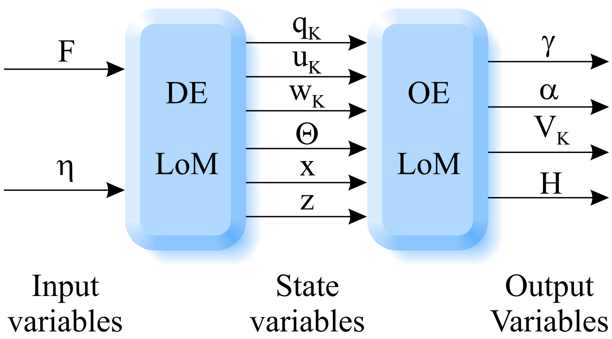


Figure 6.1: Differential equations and output equations of longitudinal motion (without wind)

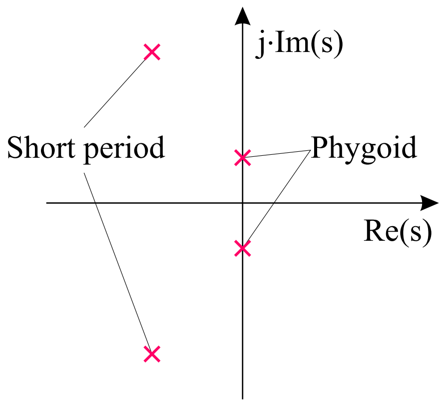


Figure 6.2: Pole distribution of the longitudinal motion

6.1.2.1 Short period

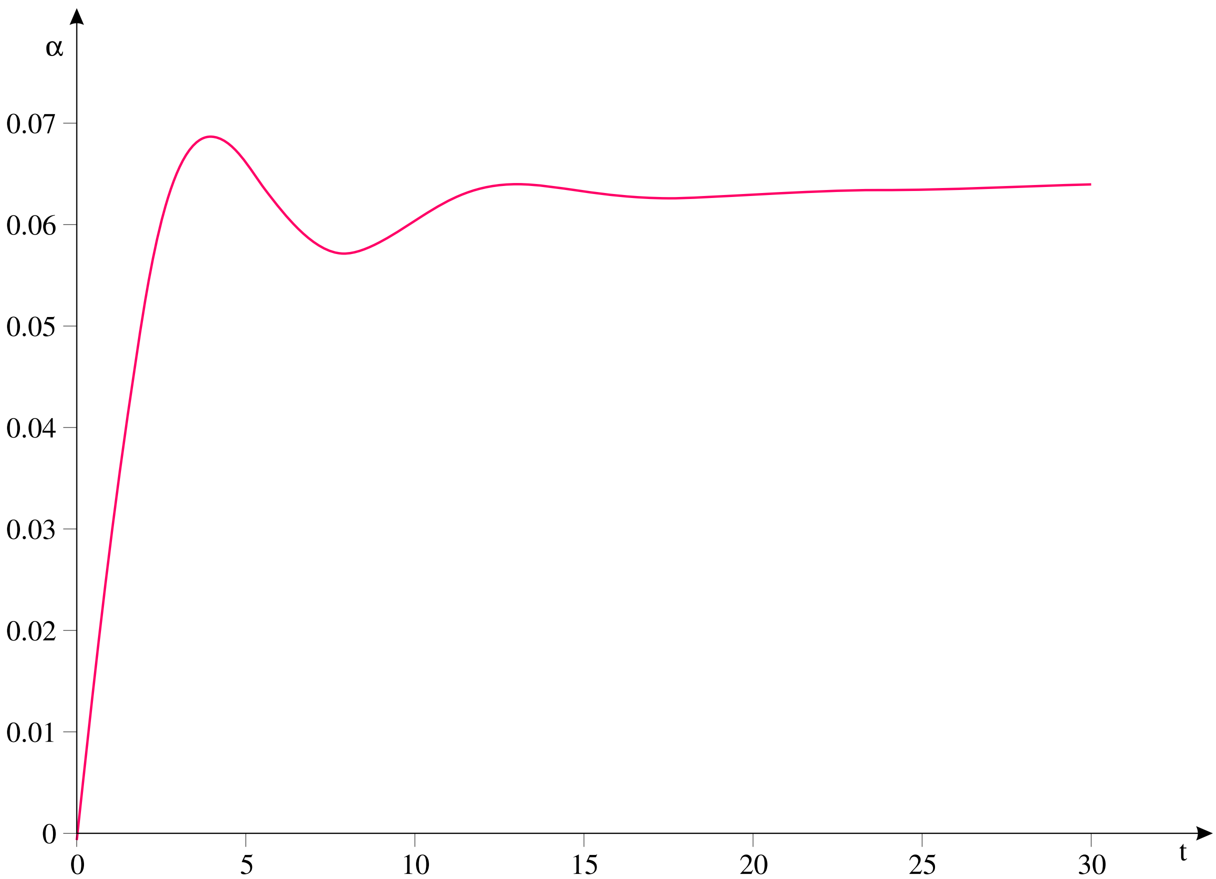


Figure 6.3: Short period

- "α oscillation"
- pitch oscillation in q_K, α, Θ
- high frequency (e. g.: $f = 0.1\,\mathrm{Hz}$, $T = 10\,\mathrm{s}$)
- medium damping (e. g.: $D = 0.5$)

6.1.2.2 Phygoid

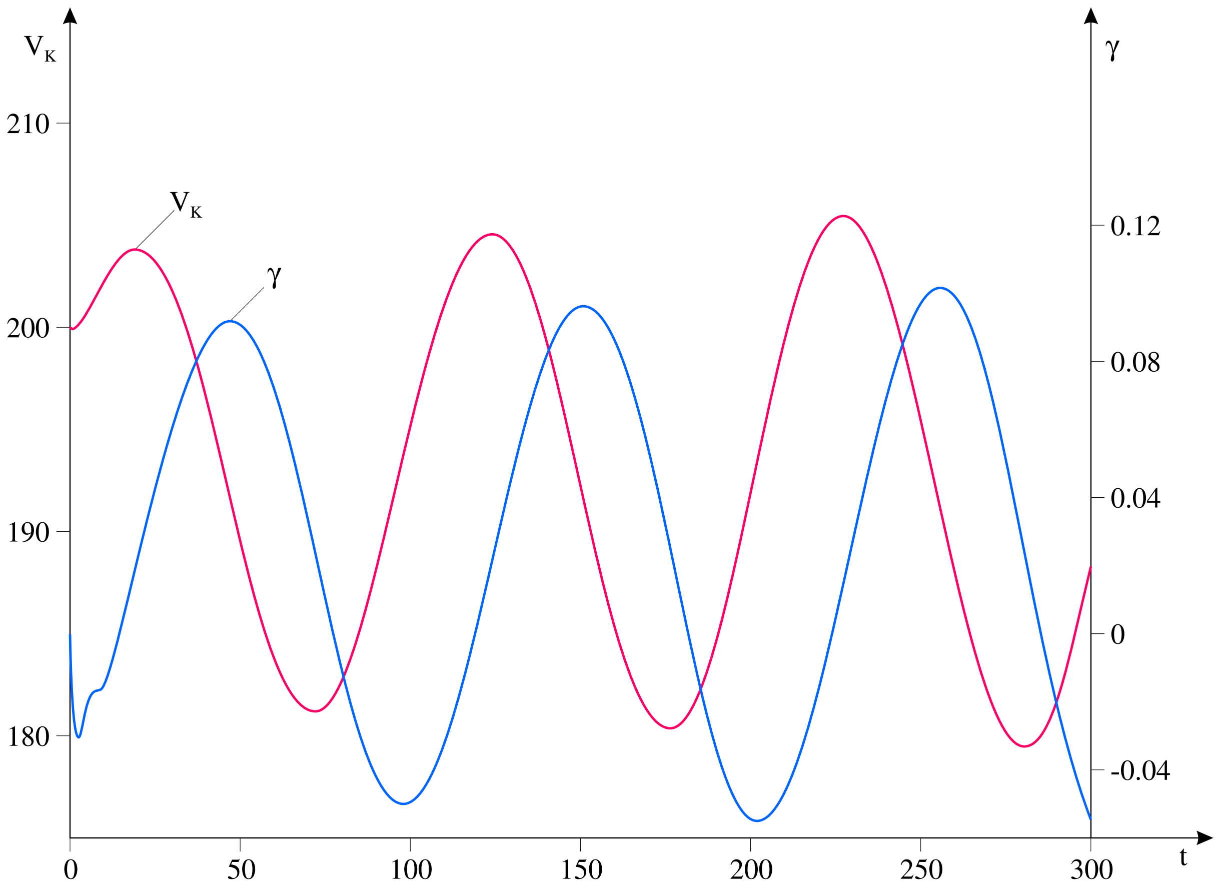


Figure 6.4: Phygoid

- "energy oscillation"
- flight-path oscillation in V_k, γ
- low frequency (e. g.: $f = 0.01\,\mathrm{Hz}$, $T = 100\,\mathrm{s}$)
- can become unstable (e. g.: $D = 0$)

6.1.3 Lateral motion

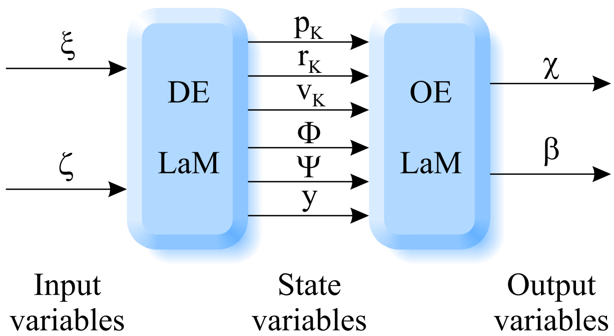


Figure 6.5: Differential equations and output equations of lateral motion (without wind)

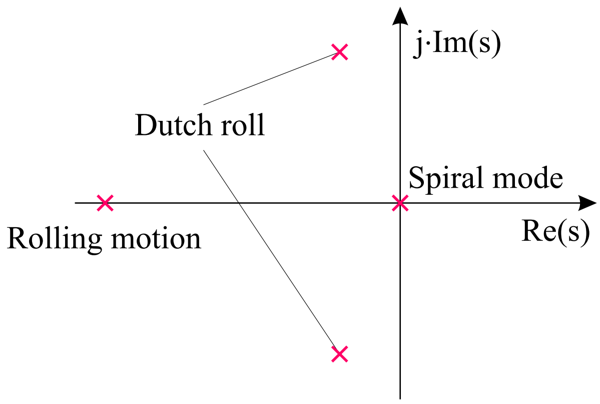


Figure 6.6: Pole distribution of the lateral motion

6.1.3.1 Dutch roll

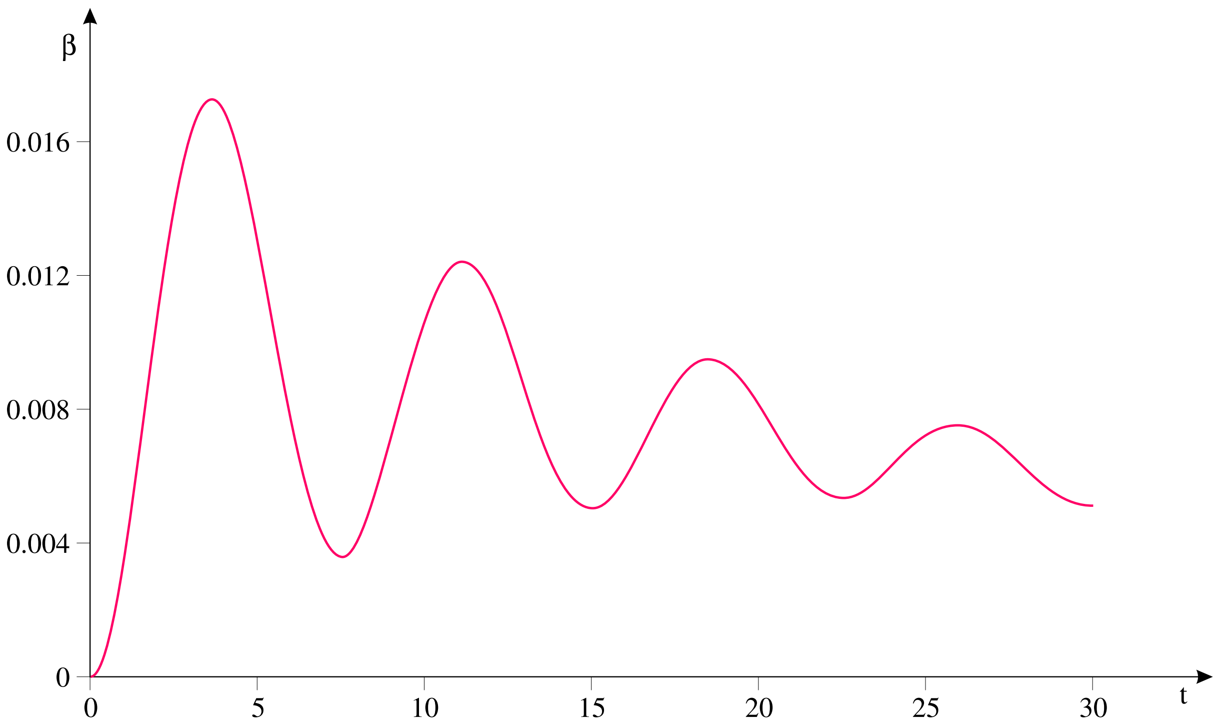


Figure 6.7: Dutch roll

- angular oscillation in β, Ψ, Φ, p_k, r_k
- high frequency (e. g.: $f = 0.1\,\mathrm{Hz}$, $T = 10\,\mathrm{s}$)
- low damping (e. g.: $D = 0.1$)

6.1.3.2 Roll motion

- "roll low pass", "roll delay"
- aperiodic roll motion in p_k, Φ
- time constant: e. g.: $T = 1\,\mathrm{s}$

6.1.3.3 Spiral mode

- "open integrator" with "some" feedback
- ξ block $\rightarrow$ stationary bank angle
- can be unstable $\rightarrow$ spiral dive

6.2 Trim calculation

Trimming is the calculation of trim variables (input and state variables) so that specified trim requirements (output and state derivative variables) are met. The trim calculation determines the (mostly steady-state) initial state of a simulation.

6.2.1 Horizontal straight flight

Task: An unaccelerated, slipping-free, horizontal straight flight without wind is to be trimmed. The relevant input, state and output variables of the longitudinal motion are:

Input variables:

$$\boldsymbol{u} = \begin{bmatrix} F & \eta \end{bmatrix}^T$$

State variables:

$$\boldsymbol{x} = \begin{bmatrix} q_{Kf} & u_{Kf} & w_{Kf} & \Theta \end{bmatrix}^T$$

Output variables:

$$\boldsymbol{v} = \begin{bmatrix} V_K & \gamma & \alpha & \cdots \end{bmatrix}^T$$

First thought: Horizontal flight is defined by the specification of a certain flight-path velocity V_K and by the requirement of a vanishing angle of climb $\gamma = 0$. These two trim requirements can be fulfilled by the two trim variables thrust F and elevator η. Thrust and elevator adjust both the energy and the moment balance of the longitudinal motion in such a way that an unaccelerated flight at a constant altitude is possible.

Second thought: Unfortunately, from a physical point of view, thrust and elevator do not directly influence the desired flight-path speed and angle of climb. Thrust and elevator generate forces and moments that directly lead only to accelerations and angular accelerations. The (angular) accelerations are then integrated by the (angular) velocity integrators of the kinetics into velocities and angular velocities. In the second integration step, position and attitude follow from this.

For example, the elevator η essentially generates a pitch moment M, which leads directly to a pitch acceleration $\dot{q}_K$ via the pitch moment of inertia. The first integration then turns the pitch acceleration into a pitch speed q_K and the second integrator generates the pitch angle Θ from this. Pitching simultaneously changes the angle of attack α, which leads to a change of the lift A. The changed lift (vertical force) then generates a vertical acceleration $\dot{w}_K$, which in turn is integrated into a vertical speed w_K. Only this vertical velocity results in a vertical position change and thus the desired change of the angle γ.

To cut a long story short: A trim algorithm cannot directly set the desired trim requirement (angle of climb) by "tweaking" the elevator trim value.

Third thought: The internal state variables involved in the definition of the desired flight state (q_{Kf}, u_{Kf}, w_{Kf}, Θ) must also be trimmed. The determination of the pitch speed q_{Kf} is simple. It must of course disappear for a steady-state straight flight, as otherwise the aircraft would pitch up or down permanently. To determine the remaining three trim values (u_{Kf}, w_{Kf}, Θ), three more trim requirements must be found. These follow directly from the requirement for an unaccelerated flight: neither pitch accelerations nor translational accelerations may occur: $\dot{q}_{Kf} = \dot{u}_{Kf} = \dot{w}_{Kf} = 0$.

If the flight condition is trimmed out, all other output variables follow automatically: the angle of attack α, for example, is calculated directly from $\alpha = \Theta - \gamma$ for pure longitudinal motion without wind.

6.2.2 Generalisation

The knowledge gained from the example of the longitudinal motion of an aircraft can be generalised. A general non-linear dynamic system can be described by a vector differential equation and an algebraic vector output equation:

Vector differential equation:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}\,(\boldsymbol{x},\, \boldsymbol{u})$$

Vector output equation:

$$\boldsymbol{v} = \boldsymbol{g}\,(\boldsymbol{x},\, \boldsymbol{u})$$

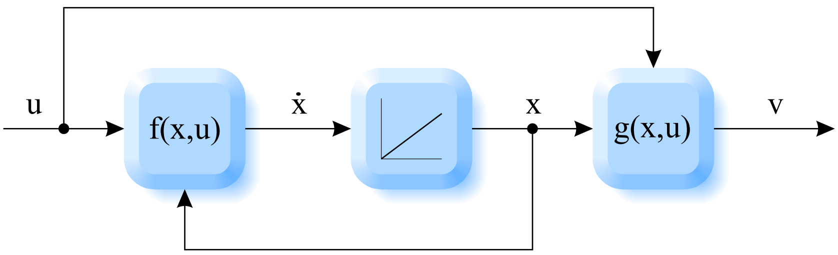


Figure 6.8: General non-linear dynamic system

Trim requirements (known)	Elements of $\dot{\boldsymbol{x}}$ and $\boldsymbol{v}$ (left side of equations)
Trim variables (unknown)	Elements of $\boldsymbol{x}$ and $\boldsymbol{u}$ (right side of equations)

6.2.2.1 Procedure

1. For each trim requirement, leave a trim variable free that can fulfill (or at least influence) the requirement.

2. For each trim variable, establish a trim requirement by which the trim variable is defined (or at least constrained).

3. Set the elements of $\boldsymbol{x}$ and $\boldsymbol{u}$, that are not trim variables to fixed values.

4. The elements of $\dot{\boldsymbol{x}}$ and $\boldsymbol{v}$, that are not trim requirements follow automatically.

6.2.3 Engine dynamics

Each dynamic subsystem (actuator dynamics, sensors, filters, controllers, ...) of the overall system to be trimmed must also be trimmed. If, for example, an engine is modeled as a limited first-order system and the thrust command F_c is searched for, which causes a desired flight-path speed V_K, then the output F of the thrust integrator must be seen as an additional trim variable and its input $\dot{F}$ as a further trim requirement. Of course, when trimmed out, the thrust F will be equal to the thrust command F_c; however, both are unknown and must therefore be determined together by the trim program.

Each additional trim variable requires exactly one additional trim requirement. Therefore, for a stationary trim point (constant thrust), the derivative of the thrust is required to be zero: $\dot{F} = 0$.

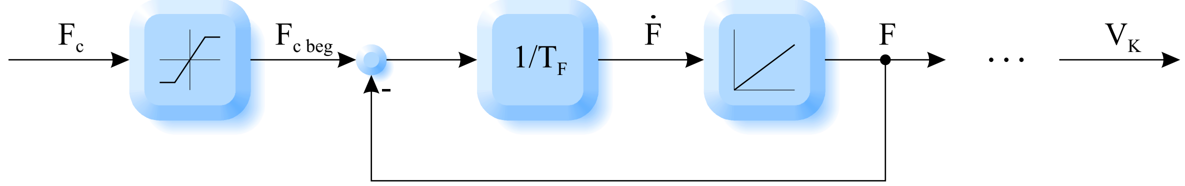


Figure 6.9: Engine dynamics with saturation

6.2.3.1 Saturations

Saturations are serious opponents for many trimming algorithms. Usually, the optimiser in the trim program varies the trim variables according to a more or less intelligent procedure until the objective function containing the trim requirements has become better than a given limit. If, however, during this search procedure a saturation is triggered because the trim algorithm has increased a trim variable above its maximum value "on a trial basis", suddenly a small variation of the trim variable no longer causes any change in the trim requirements. The gradient-oriented trim algorithm "no longer knows in which direction it should continue to optimise" and aborts with an error message.

In these cases, it often helps to reduce the maximum step size of the corresponding trim variable so that the algorithm does not "accidentally" violate the saturation in a step that is too large. In addition, it is of course useful to set each trim variable to an estimated initial value that is as close as possible to the expected trim point. For example, if the maximum thrust of an engine in cruise flight is 90 kN, it certainly makes more sense to set the initial value of the trim variable thrust to perhaps 60 kN rather than to 0 kN or even 100 kN.

6.3 Basic controller

The basic controller dampens the natural response (pitch, yaw and roll dampers) and controls the airspeed, pitch angle, bank angle and sideslip angle. The commands for these input variables of the basic controller can either come directly from the pilot (e. g. sidestick commanded *rate command attitude hold* for pitch and roll angle) or from a superimposed flight-path control loop.

6.3.1 Basic controller of the longitudinal motion

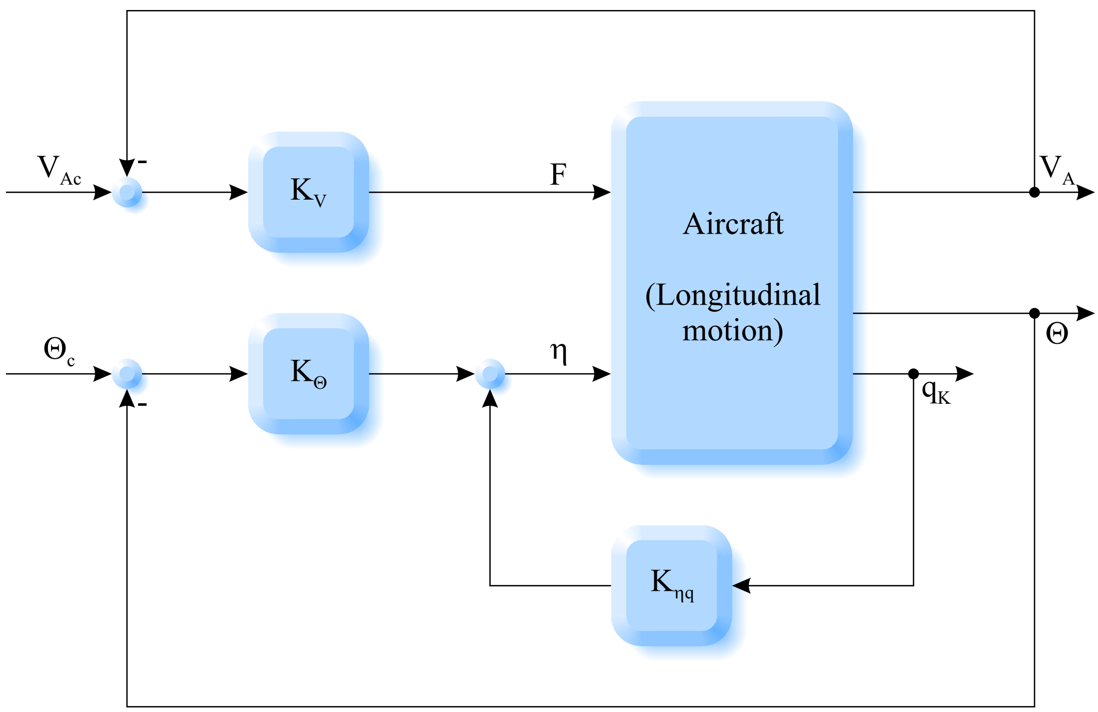


Figure 6.10: Basic controller of the longitudinal motion

$K_{\eta q}$ The **pitch damper** uses the elevator to dampen the short period. In doing so, the (measured) pitch speed is compared with a reference value of zero (so to speak), and the elevator is always deflected in such a way that the resulting pitch moment and the resulting pitch acceleration counteract the pitch speed.

K_Θ The **pitch controller** uses the same control input (elevator) as the pitch damper and controls the pitch angle. The pitch angle reference can be set directly by the pilot (e. g. by means of the longitudinal sidestick) or by an outer (cascaded) flight-path controller (altitude controller, . . .).

K_V The **airspeed controller** (autothrottle) uses the thrust to maintain an airspeed reference. To prevent a steady-state control error, the airspeed controller can have an integral component.

6.3.2 Basic controller of the lateral motion

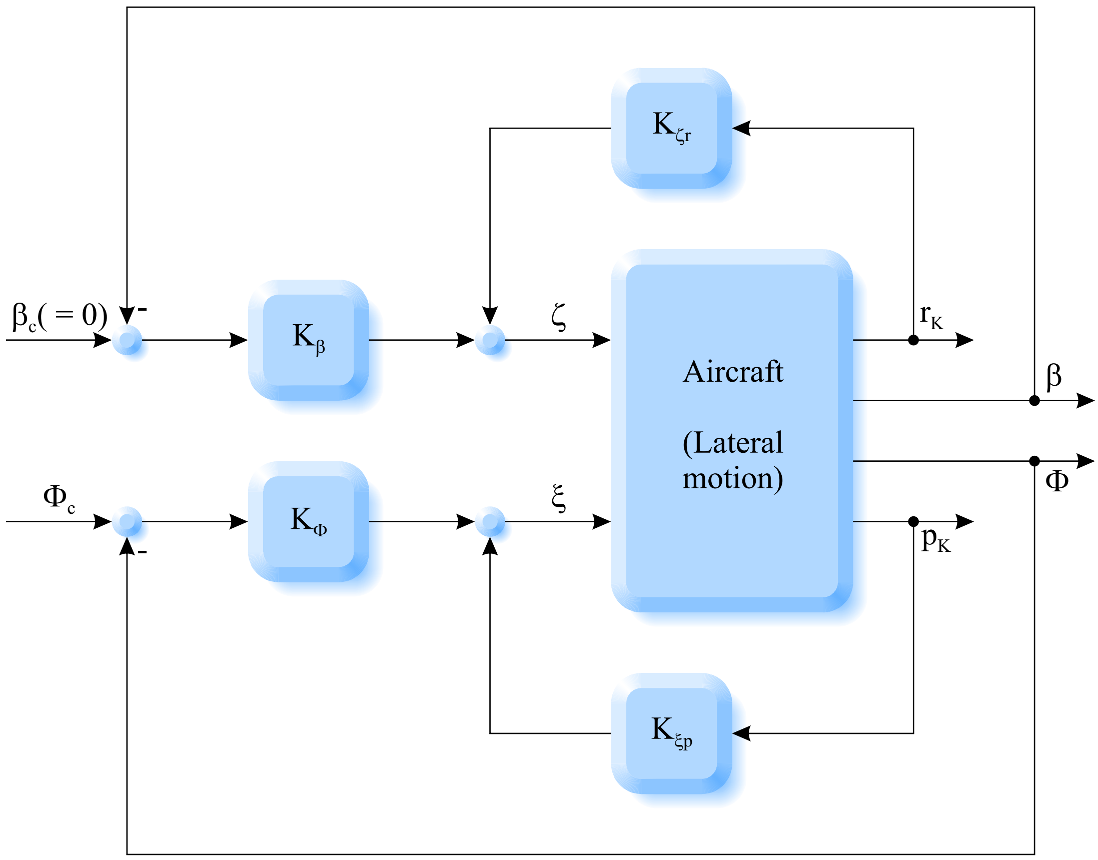


Figure 6.11: Basic controller of lateral motion

$K_{\zeta r}$ The **yaw damper** measures the yaw rate and uses the rudder to dampen the Dutch roll. During a stationary turn, the yaw damper should not be active, if possible, so as not to suppress the yaw rate that is then desired ($\rightarrow$ use of a high pass, which does not "let through" constant yaw rates).

$K_{\xi p}$ The **roll damper** uses the aileron to change the roll time constant.

K_β The **sideslip angle controller** uses the rudder to control the sideslip angle. Often, for example in a coordinated turn, the sideslip angle should disappear (reference value equal to zero) to ensure a symmetrical, economical flow. The sideslip angle reference can be commanded with the pedals, for example, or it is specified by the autopilot (e. g. during the decrab manoeuvre to align the landing gear in the direction of the runway when approaching with a crosswind).

K_Φ The **bank angle controller** uses the same control input (aileron) as the roll damper and provides for the control of the bank angle. The desired bank angle can be set directly by the pilot (e. g. by means of the lateral sidestick) or by an outer (cascaded) flight-path controller controller (course controller, . . .).

6.4 Flight-path controller

The flight-path controller (altitude controller and flight-path azimuth controller) uses the "basic-controlled" aircraft as a "modified plant" in the sense of a cascade control. The control variables of the basic controller (pitch angle and bank angle) are directly commanded by the flight-path controller.

6.4.1 Cascade control

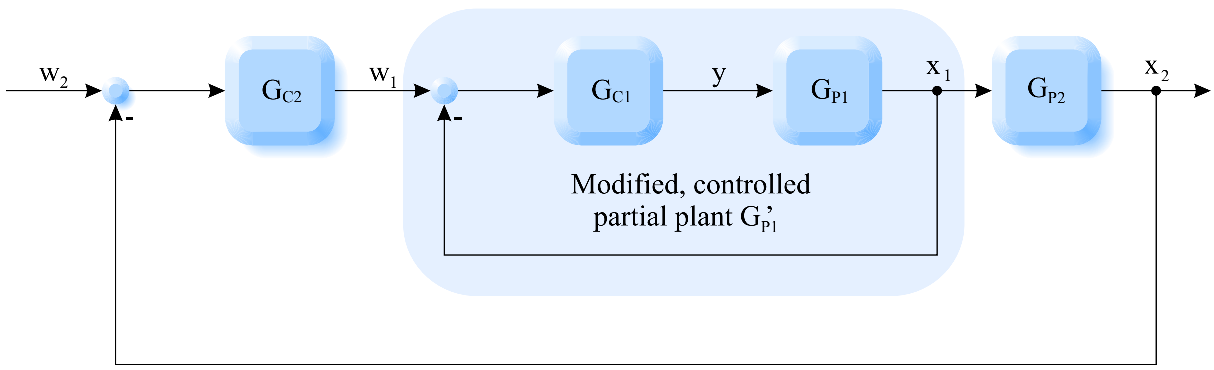


Figure 6.12: Cascade control

- Outer control loop supplies reference for inner control loop.
- Design the inner control loop to be fast and without steady-state accuracy.
- Outer control loop provides steady-state accuracy.

6.4.2 Altitude and flight-path azimuth control

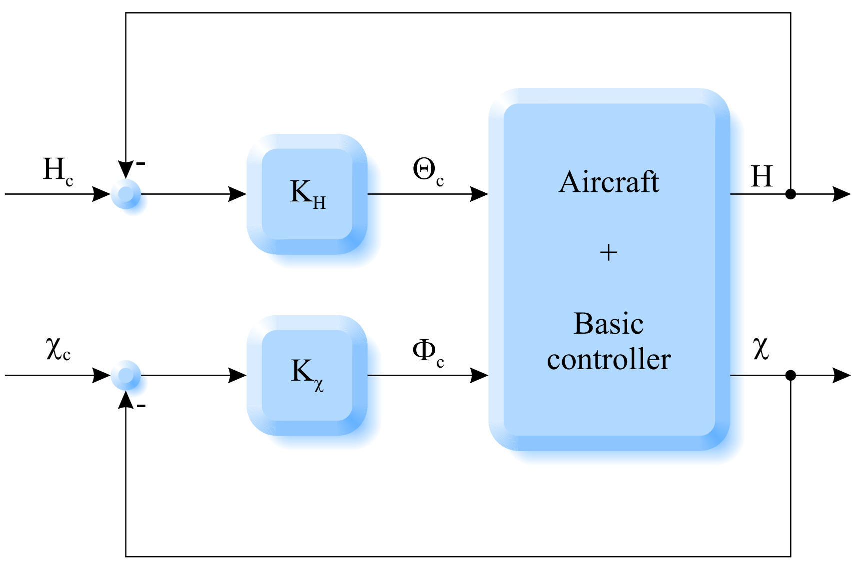


Figure 6.13: Altitude and flight-path azimuth control

K_H In this cascade control, the **altitude controller** does not have direct access to the elevator, but commands a pitch angle reference to the basic controller of the longitudinal motion, which in turn intervenes in the aircraft's moment balance via the elevator in order to adjust the pitch angle accordingly. In order to achieve steady-state accuracy for altitude control, the altitude controller can be designed as a PI(D) controller.

K_χ The **flight-path azimuth controller** controls the flight-path azimuth by passing on a reference for the bank angle to the basic controller. As an alternative to the flight-path azimuth, the yaw angle (heading) can be used as a controlled variable (e. g. for availability reasons).

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